

SYMMETRIES AND MASS PREDICTIONS OF THE SUPERSYMMETRIC QUARK MODEL

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Abstract

QCD justification of $SU(m/n)$ supergroups are shown to provide a basis for the existence of an approximate hadronic supersymmetry. Effective Hamiltonian of the relativistic quark model is derived, leading to hadronic mass formulae in remarkable agreement with experiments. Bilocal approximation to hadronic structure and incorporation of color through octonion algebra (based on quark-antiquark symmetry) is also shown to predict exotic diquark-antidiquark ($D-\bar{D}$) meson states. A minimal supersymmetric scheme based on $SU(3)^c \times SU(6/1)$ that excludes exotics is constructed. Symmetries of three quark systems and possible relativistic formulation of the quark model through the spin realization of Wess Zumino algebra is presented.

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Introduction

In 1966 Miyazawa, in a series of papers^[1], extended the $SU(6)$ group to the supergroup $SU(6/21)$ that could be generated by constituent quarks and diquarks that could be transformed to each other. In particular, he found the following

- a general definition of $SU(m/n)$ superalgebras, expressing the symmetry between m bosons and n fermions, with Grassman-valued parameters.
- a derivation of the super-Jacobi identity.
- the relation of the baryon mass splitting to the meson mass splitting through the new mass formulae.

This work contained the first classification of superalgebras (later rediscovered by mathematicians in the seventies). Because of the field-theoretic prejudice against $SU(6)$, Miyazawa's work was generally ignored. Supersymmetry was, of course, rediscovered in the seventies within the dual resonance model by Ramond^[2], Neveu and Schwarz^[3] (1971). Golfand and Likhtman^[4], and independently Akulov and Volkov^[5], proposed the extension of the Poincaré group to the super-Poincaré group. Examples of supersymmetric field theories were given and the general method based on the super-Poincaré group was discovered by Wess and Zumino^[6] (1974). The super-Poincaré group allowed transformations between fields associated with different spins 0, $\frac{1}{2}$ and 1. The Coleman-Mandula theorem was amended in 1975 by Haag, Lopuszanski and Sohnius to allow **super – Poincaré group** \times **G_{int}** as the maximum symmetry of the S-matrix. Unfortunately, $SU(6)$ symmetry was still forbidden.

$SU(6)$ and Hadronic Supersymmetry

How do we interpret the symmetries of the QCD spectrum in this light? In the ultraviolet, the running coupling constant tends to zero and quarks behave like free point particles. Thus an approximate conformal symmetry exists, allowing spin to be conserved separately from orbital angular momentum. Thus spin behaves as an internal quantum number; this makes an $SU(6)$ symmetry possible, since the quarks are almost free Dirac particles. Single vector-gluon exchange breaks this symmetry; thus, as shown by

Glashow, Georgi and deRujula^[7], the mass-degeneracy of hadrons of different spins is lifted by a hyperfine-interaction term.

Here is the main point. In the infrared we expect confinement to set in. The quark-antiquark potential becomes proportional to the distance. Careful studies of quarkonium spectra and lattice-gauge calculations show that at large separation the quark forces become spin-independent. QCD is also flavor independent. We therefore find approximate spin- and flavor-independent quark binding forces; these are completely consistent with $SU(6)$ symmetry. This is not an exact symmetry, but is a good starting point, before spin and flavor effects are included.

There is good phenomenological evidence that in a rotationally excited baryon a quark-diquark ($q-D$) structure is favored over a three-quark (qqq) structure^{[8],[9],[10]}. Eguchi^[11] had shown that it is energetically favorable for the three quarks in a baryon to form a linear structure with a quark on one end and bilocal structure qq at the other end. Similarly, Johnson and Thorn^[12] had shown that the string-like hadrons may be pictured as vortices of color flux lines which terminate on concentration of color at the end points. A baryon with three valence quarks would be arranged as a linear chain of molecule where the largest angular momentum for a state of a given mass is expected when two quarks are at one end, and the third is at the other: At large spin, two of the quarks form a diquark at one end of the string, the remaining quark being at the other. Regge trajectories for mesons and baryons are closely parallel; both have a slope of about $0.9(GeV)^{-2}$. If the quarks are light, the underlying quark-diquark symmetry leads to a Miyazawa symmetry between mesons and baryons. Thus we studied QCD with a weakly broken supergroup $SU(6/21)$. Note that the fundamental theory is not supersymmetric. For quarks, the generators of the Poincaré group and those of the color group $SU(3)^c$ commute. It is only the effective Hamiltonian which exhibits an approximate supersymmetry among the bound states $q\bar{q}$ and qD .

Under the color group $SU^C(3)$, meson $q\bar{q}$ and diquark ($D = qq$) states transform as^{[10],[13]}

$$qq : \mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \quad ; \quad q\bar{q} : \mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \quad (1)$$

and under the spin flavor $SU(6)$ they transform as

$$qq : \mathbf{6} \times \mathbf{6} = \mathbf{15} + \mathbf{21} \quad ; \quad q\bar{q} : \mathbf{6} \times \bar{\mathbf{6}} = \mathbf{1} + \mathbf{35} \quad (2)$$

Dimensions of internal degrees of quarks and diquarks are shown in the following table:

	$SU_f(3)$	$SU_s(2)$	$dim.$
q	\square	$s = 1/2$	$3 \times 2 = 6$
D	$\square \quad \square$	$s = 1$	$6 \times 3 = 18$
	\square	$s = 0$	$3 \times 1 = 3$
	\square		

If one writes qqq as qD , then the quantum numbers of D are $\bar{\mathbf{3}}$ for color since when combined with q must give a color singlet, and $\mathbf{21}$ for spin-flavor since combined with color must give antisymmetric wavefunctions. The quantum numbers for \bar{q} are for color, $\bar{\mathbf{3}}$, and for spin-flavor, $\bar{\mathbf{6}}$. Thus \bar{q} and D have the same quantum numbers (color forces can not distinguish between \bar{q} and D). Therefore there is a dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix}, \quad \bar{\psi} = (q \quad \bar{D}) \quad (3)$$

We can obtain all hadrons by combining ψ and $\bar{\psi}$: mesons are $q\bar{q}$, baryons are qD , and exotics are $D\bar{D}$ states. Inside rotationally excited baryons, QCD leads to the formation of diquarks well separated from the remaining quark. At this separation the scalar, spin-independent, confining part of the effective QCD potential is dominant. Since QCD forces are also flavor-independent, the force between the quark q and the diquark D inside an excited baryon is essentially the same as the one between q and the antiquark \bar{q} inside an excited meson. Thus the approximate spin-flavor independence of hadronic physics expressed by $SU(6)$ symmetry is extended to $SU(6/21)$ supersymmetry through a symmetry between \bar{q} and D , resulting in parallelism of mesonic and baryonic Regge trajectories.

QCD Justification of $U(6/21)$ Supersymmetry and Its Breaking

The approximate group $SU(6)$ and its supersymmetric extension to $SU(6/21)$ was justified within the standard theory of colored quarks interacting through gluons that are associated with a color group $SU(3)^c$. The justification of $SU(6)$ and the derivation of its breaking was given by Georgi, Glashow and de Rujula^[7]. We extended their approach through use of supergroups of type $SU(m/n)$ which provided a basis for the existence of an approximate hadronic supersymmetry^{[10],[13]}.

We shall first discuss the validity domain of $SU(6/21)$ supersymmetry. The diquark structure with spin $s = 0$ and $s = 1$ emerges in inelastic inclusive lepton-baryon collisions with high impact parameters that excite the baryon rotationally, resulting in inelastic structure functions based on point-like quarks and diquarks instead of three point-like quarks. In this case both mesons and baryons are bilocal with large separation of constituents.

Also, there is a symmetry between color antitriplet diquarks with $s = 0$ and $s = 1$ and color antitriplet antiquarks with $s = \frac{1}{2}$. This is only possible if the force between quark q and antiquark \bar{q} , and also between q and diquark D is mediated by a zero spin object that sees no difference between the spins of \bar{q} and D . The object can be in color states that are either singlet or octet since q and D are both triplets. Such an object is provided by scalar flux tubes of gluons that dominate over the one gluon exchange at large distances. Various strong coupling approximations to QCD, like lattice gauge theory^{[14],[15]}, 't Hooft's $\frac{1}{N}$ approximation^[16] when N , the number of colors, is very large, or the elongated bag model^[12] all give a linear potential between widely separated quarks and an effective string that approximates the gluon flux tube. In such a theory it is energetically favorable for the three quarks in a baryon to form a linear structure with a quark in the middle and two at the ends, or, for high rotational excitation, a bilocal linear structure (diquark) at one end and a quark at the other end. In order to illustrate these points we start with the suggestion of Johnson and Thorn^[12] that the string-like hadrons may be pictured as vortices of color flux lines which terminate on concentration of color at the end points. The color flux connecting opposite ends is the same for mesons and baryons giving an explanation for the same slope of meson and baryon trajectories^[10].

To construct a solution which yields a maximal angular momentum for a fixed mass we consider a bag with elongated shape rotating about the center of mass with an angular frequency ω . Its ends have the maximal velocity

allowed, which is the speed of light ($c = 1$). Thus, a given point inside the bag, at a distance r from the axis of rotation moves with a velocity

$$v = \vec{\omega} \cdot \vec{r} = \frac{2r}{L} \quad (4)$$

where L is the length of the string. In this picture the bag surface will be fixed by balancing the gluon field pressure against the confining vacuum pressure B , which (in analogy to electrodynamics) can be written in the form

$$\frac{1}{2} \sum_{\alpha=1}^8 (E_{\alpha}^2 - B_{\alpha}^2) = B \quad (5)$$

Using Gauss' law the color electric field E through the flux tube connecting the color charges at the ends of the string is given by

$$\int \vec{E}_{\alpha} \cdot d\vec{S} = E_{\alpha} A = g \frac{1}{2} \lambda_{\alpha} \quad (6)$$

where $A(r)$ is the cross-section of the flux tube at distance r from the center and $g \frac{1}{2}$ is the color electric charge which is the source of E_{α} . By analogy with classical electrodynamics the color magnetic field $\vec{B}_{\alpha}(r)$ associated with the rotation of the color electric field is

$$\vec{B}_{\alpha}(r) = \vec{v}(r) \times \vec{E}_{\alpha}(r) \quad (7)$$

at a point moving with a velocity $\vec{v}(r)$. For the absolute values this implies

$$B_{\alpha} = v E_{\alpha} \quad (8)$$

because $\vec{v}(r)$ is perpendicular to $\vec{E}_{\alpha}(r)$. Using last three equations together with

$$\langle \sum_{\alpha=1}^8 (\frac{1}{2} \lambda_{\alpha})^2 \rangle = \frac{4}{3} \quad (9)$$

for the $SU(3)^c$ triplet in Eq.(5) we obtain for the crosssection of the bag

$$A(r) = \sqrt{\frac{2}{3B}} g \sqrt{1 - v^2} \quad (10)$$

which shows the expected Lorentz contraction.

The total energy E of the bag

$$E = E_q + E_G + BV \quad (11)$$

is the sum of the quark energy E_q , the gluon field energy E_G and the volume energy of the bag, BV . Because the quarks at the ends move with the a speed close to speed of light, their energy is simply given by

$$E_q = 2p \quad (12)$$

where p is the momentum of a quark, a diquark or an antiquark, respectively. For the gluon energy, by analogy with electrodynamics, one obtains from Eqs.(6-8) the result

$$\begin{aligned} E_G &= \frac{1}{2} \int d^3x \sum_{\alpha=1}^8 (E_\alpha^2 + B_\alpha^2) \\ &= \sqrt{\frac{2}{3}} g \sqrt{BL} \int_0^1 dv \frac{1+v^2}{\sqrt{1-v^2}} = \sqrt{\frac{2}{3}} g \sqrt{BL} \frac{3\pi}{4} \end{aligned} \quad (13)$$

and for the volume energy

$$\begin{aligned} BV &= 2B \int_0^{\frac{L}{2}} A(r) dr = 2B \int_0^1 \sqrt{\frac{2}{3B}} g \sqrt{1-v^2} \frac{L}{2} dv \\ &= \sqrt{\frac{2}{3}} g \sqrt{BL} \frac{\pi}{4} = \frac{BA(0)L\pi}{4} \end{aligned} \quad (14)$$

It is obvious from Eq.(13) that the gluon field energy is proportional to the length L of the bag. The gluon field energy and the volume energy of the bag together correspond to a linear rising potential of the form

$$V(L) = E_G + BV = bL \quad (15)$$

where

$$b = \sqrt{\frac{2B}{3}} g \pi \quad (16)$$

The total angular momentum J of this classical bag is the sum of the angular momenta of the quarks at the two ends

$$J_q = pL \quad (17)$$

and the angular momentum J_G of the gluon field. From Eq.(7) we get

$$\vec{E}_\alpha \times \vec{B}_\alpha = \vec{v} E_\alpha^2 \quad (18)$$

for the momentum of the gluon field, and hence

$$\begin{aligned} J_G &= \left| \int_{bag} d^3\vec{r} \sum_{\alpha=1}^8 \vec{r} \times (\vec{E}_\alpha \times \vec{B}_\alpha) \right| = 2 \int_0^{\frac{L}{2}} dr A(r) r v E_\alpha^2 \\ &= \frac{16}{3L} g^2 \int_0^{\frac{L}{2}} \frac{r^2 dr}{A(r)} = \sqrt{\frac{2}{3}} g \sqrt{B} L^2 \frac{\pi}{4} \end{aligned} \quad (19)$$

where we have used Eq.(4) and Eq.(6) in the third step. We can now express the total energy of the bag in terms of angular momenta. Putting these results back into expressions for E_q and E_G , we arrive at

$$E_q = \frac{2J_q}{L}, \quad E_G = \frac{3J_G}{L} \quad (20)$$

so that the bag energy now becomes

$$\begin{aligned} E &= \frac{2J_q}{L} + \frac{3J_G}{L} + \sqrt{\frac{2B}{3}} L g \frac{\pi}{L} = \frac{2J_q + 4J_G}{L} \\ &= \frac{2(J + J_G)}{L} = \frac{1}{L} (2J + \sqrt{\frac{2}{3}} g \sqrt{B} L^2 \frac{\pi}{2}) \end{aligned} \quad (21)$$

Minimizing the total energy for a fixed angular momentum with respect to the length of the bag, $\frac{dE}{dt} = 0$ gives the relation

$$-\frac{2J}{L^2} + \sqrt{\frac{2}{3}} g \sqrt{B} \frac{\pi}{2} = 0 \quad (22)$$

so that

$$L^2 = \frac{4J}{g\pi} \sqrt{\frac{3}{2B}} \quad (23)$$

Re-inserting this into Eq.(21) we arrive at

$$E = 2\sqrt{Jg\pi}\left(\frac{2B}{3}\right)^{\frac{1}{4}} \quad (24)$$

or

$$J = \left(\sqrt{\frac{3}{2B}} \frac{1}{4\pi g}\right) E^2 = \left(\sqrt{\frac{3}{2B}} \frac{1}{8\pi^{\frac{3}{2}}} \frac{1}{\sqrt{\alpha_s}}\right) E^2 = \alpha'(0) M^2 \quad (25)$$

where $M = E$ and we used $\alpha_s = \frac{g^2}{4\pi}$, the unrationalized color gluon coupling constant. We can now let $\alpha'(0)$ defined by the last equation which is the slope of the Regge trajectory as

$$\alpha'(0) = \sqrt{\frac{3}{2B}} \frac{1}{8\pi^{\frac{3}{2}}} \frac{1}{\sqrt{\alpha_s}} = \frac{1}{4b} \quad (26)$$

where b was defined in Eq.(15).

The parameters B and α_s have been determined^{[17].[18]} using the experimental information from the low lying hadron states: $B^{\frac{1}{4}} = 0.146 \text{ GeV}$ and $\alpha_s = 0.55 \text{ GeV}$. If we use these values in Eq.(26) we find

$$\alpha'(0) = 0.88 \text{ (GeV)}^{-2} \quad (27)$$

in remarkable agreement with the slope determined from experimental data which is about 0.9 (GeV)^{-2} .

The total phenomenological non-relativistic potential then is the well known superposition of the Coulomb-like and confining potentials $V(r) = \frac{a}{r} + br$ where $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between q and \bar{q} in a meson, or between q and D in a baryon with high angular momentum. This is verified in lattice QCD to a high degree of accuracy^[19] ($a = \frac{-c\alpha_c}{r}$, where c is the color factor and α_c the strong coupling strength).

It is interesting to know that all this is related very closely to the dual strings. Indeed we can show that the slope given in Eq.(26) is equivalent to the dual string model formula for the slope if we associate the "proper tension" in the string with the proper energy per unit length of the color flux tube and the volume. By proper energy per unit length we mean the energy per unit length at a point in the bag evaluated in the rest system of that point. This will be

$$T_0 = \frac{1}{2} \sum_{\alpha} E_{\alpha}^2 A_0 + B A_0 \quad (28)$$

The fact that $\frac{1}{2} \sum_{\alpha} E_{\alpha}^2 = B$ in the rest sytem gives

$$T_0 = 2BA_0 \quad (29)$$

where A_0 is the cross-sectional area of the bag. If in Eq.(10) we let $A = A_0$ when $v = 0$, then using

$$A_0 = \sqrt{\frac{2}{3B}}g \quad (30)$$

we find

$$T_0 = 2\sqrt{\frac{2}{3}}g\sqrt{B} = 4\sqrt{\frac{2\pi}{3}}\sqrt{\alpha_s}\sqrt{B} \quad (31)$$

for the proper tension. In the dual string the slope and proper tension are related by the formula^[20]

$$T_0 = \frac{1}{2\pi\alpha'} \quad (32)$$

so the slope is

$$\alpha' = \frac{1}{8}\sqrt{\frac{3}{2}}\frac{1}{\pi^{\frac{3}{2}}}\frac{1}{\sqrt{\alpha_s}}\frac{1}{\sqrt{B}} \quad (33)$$

which is identical to the earlier formula we produced in Eq.(26).

It would appear from Eq.(31) that the ratio of volume to field energy would be one-to-one in one space dimension in contrast to the result one-to-three which holds for a three dimensional bag^[21]. However, the ratio one-to-one is true only in the rest system at a point in the bag, and each position along the x -axis is of course moving with a different velocity. Indeed we see from Eq.(13) and Eq.(14) that the ratio of the total volume energy to the total field energy is given by one-to three in conformity with the virial theorem^[21].

In the string model of hadrons we have $E^2 \sim J$ between the energy and the angular momentum of the rotating string. If we denote by $\rho(r)$ the mass density of the string, and by v and ω its linear and angular velocities, respectively, the energy and the angular momentum of the rotating string are given by

$$E = 2 \int \frac{\rho(r)}{\sqrt{1 - \omega^2 r^2}} dr = \frac{2}{\omega} \int_0^1 \frac{\rho(v)}{\sqrt{1 - v^2}} dv \quad (34)$$

and

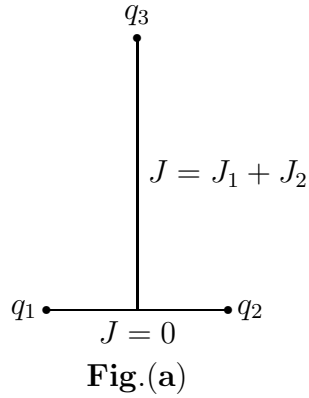
$$J = 2 \int \frac{\rho(r)}{\sqrt{1 - \omega^2 r^2}} r^2 \omega dr = \frac{2}{\omega^2} \int_0^1 \frac{\rho(v)}{\sqrt{1 - v^2}} v^2 dv \quad (35)$$

and hence the relation

$$E^2 \propto J. \quad (36)$$

If the string is loaded with mass points at its ends, they no longer move with speed of light, however, the above relation still holds approximately for the total energy and angular momentum of the loaded string.

We now look at various ways of partitioning of the total angular momentum into two subsystems. Figures (a), (b) and (c) show the possible configurations of three quarks in a baryon.



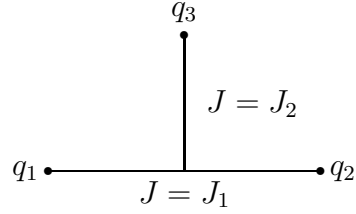


Fig.(b)

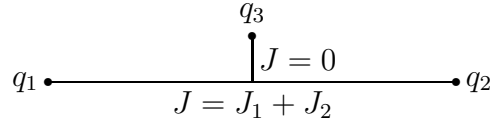


Fig.(c)

If we put the proportionality constant in Eq.(36) equal to unity, then the naive evaluation of energies yield

$$E^2 = J_1 + J_2 = E_1^2 + E_2^2 \leq (E_1 + E_2)^2 = E'^2 \quad (37)$$

where E and E' denote the energies corresponding to figures (a) or (c). In the case of figure (b), J_1 , J_2 are the angular momenta corresponding to the energies E_1 and E_2 of the subsystems. The equality in Eq.(37) holds only if E_1 or E_2 is zero. Therefore for each fixed total angular momentum its most unfair partition into two subsystems gives us the lowest energy levels, and its more or less fair partition gives rise to energy levels on daughter trajectories.

Hence on the leading baryonic trajectory we have a quark-diquark structure (Fig.(a)), or a linear molecule structure (Fig.(c)). On the other hand on low-lying trajectories we have more or less symmetric ($J_1 \sim J_2$) configuration of quarks.

Since high J hadronic states on leading Regge trajectories tend to be bilocal with large separation of their constituents, they fulfill all the conditions for supersymmetry between \bar{q} and D . Then the only difference between the energies of $(q\bar{q})$ mesons and (qD) baryons comes from the different masses of their constituents, namely $m_q = m_{\bar{q}} = m$, and $m_D \sim 2m$. For high J this is the main source of symmetry breaking which is spin independent. We will show how we can obtain sum rules from this breaking. The part of the mass operator that gives rise to this splitting is a diagonal element of $U(6/21)$ that commutes with $SU(6)$.

Let us now consider the spin dependent breaking of $U(6/21)$. For low J states the (qD) system becomes trilocal(qqq), the flux tube degenerates to a single gluon propagator that gives spin-dependent forces in addition to the Coulomb term $\frac{a}{r}$. In this case we have the regime studied by de Rujula, Georgi and Glashow, where the breaking is due to hyperfine splitting caused by the exchange of single gluons that have spin 1. These mass splittings give rise to different intercept of the Regge trajectories are given by

$$\Delta m_{12} = k \frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2}, \quad k = |\psi(0)|^2 \quad (38)$$

both for baryons and mesons at high energies. But at low energies the baryon becomes a trilocal object (with three quarks) and the mass splitting is given by^[22]

$$\Delta m_{123} = \frac{1}{2} k \left(\frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2} + \frac{\vec{S}_2 \cdot \vec{S}_3}{m_2 m_3} + \frac{\vec{S}_3 \cdot \vec{S}_1}{m_3 m_1} \right) \quad (39)$$

where m_1 , m_2 and m_3 are the masses of the three different quark constituents.

The element of $U(6/21)$ that give rise to such splittings is a diagonal element of its $U(21)$ subgroup and gives rise to $s(s+1)$ terms that behave like an element of the (405) representation of $SU(6)$ in the $SU(6)$ mass formulae. The splitting of isospin multiplets is due to a symmetry breaking element in the (35) representation of $SU(6)$. hence all symmetry breaking terms are in the adjoint representation of $U(6/21)$. If we restrict ourselves to the non-strange sector of hadrons with approximate $SU(4)$ symmetry, effective

supersymmetry will relate the splitting in m^2 between the Δ ($s = \frac{3}{2}, I = \frac{3}{2}$) and N ($s = \frac{1}{2}, I = \frac{1}{2}$) to the splitting between ω ($s = 1, I = 0$) and π ($s = 0, I = 1$) so that

$$m_{\Delta}^2 - m_N^2 \simeq m_{\omega}^2 - m_{\pi}^2 \quad (40)$$

which is satisfied to within 5%.

Effective Hamiltonian of the Relativistic Quark Model

The quark model is considered to be an effective theory of QCD at low energies when the gluon degrees of freedom are integrated out. A relativistic potential model of quarks can be built based upon the effective color potentials. Lattice gauge calculations indicate that, inside a hadron, quarks are bounded by a confining potential that grows like r , where r is the separation between two quarks, for large r . At small r , the color force is coulomb-like, i.e. $\sim 1/r$.

Following the treatment given in our previous papers^{[10],[13]} we consider a quark $q(1)$ at position \mathbf{r}_1 , and an antiquark $\bar{q}(2)$ at position \mathbf{r}_2 represented by the charge conjugate spinor $q^c(2)$. The $q(1)\bar{q}(2)$ system for a meson has 16 space components that are the elements of the 4×4 matrix

$$\Phi(1, 2) = q(1)\bar{q}(2). \quad (41)$$

Elimination of gluon degrees of freedom generates a potential between these constituents that in the static approximation depends on the relative distance r . It will be approximated by a scalar potential $S(r)$ and the fourth component of a vector potential $V(r)$ given by

$$S(r) = br, \quad (42)$$

$$V(r) = -\frac{4}{3}\alpha_s \frac{1}{r}, \quad (43)$$

where α_s is the strong coupling constant at an energy scale corresponding to low lying meson masses. In this approximation the relativistic Hamiltonian can be written as

$$H\Phi = (H_{(1)} + H_{(2)})\Phi, \quad (44)$$

with

$$H_{(i)} = \frac{1}{2}V(r) + \gamma_4^{(i)}[m_i + \frac{1}{2}S(r)] + i\gamma_4^{(i)}\boldsymbol{\gamma}^{(i)}\cdot\mathbf{p}^{(i)}, \quad (45)$$

where $i = 1, 2$, and $\mathbf{p}^{(i)} = -i\nabla^{(i)}$. At this point let us introduce the center of mass and relative coordinates

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad (46)$$

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}, \quad (47)$$

we have

$$\mathbf{p}_1 = m_1\dot{\mathbf{R}} + \mu\dot{\mathbf{r}}, \quad \mathbf{p}_2 = m_2\dot{\mathbf{R}} - \mu\dot{\mathbf{r}}, \quad (48)$$

where μ is the reduced mass. In the center of mass system ($\mathbf{R} = 0$), we find

$$\mathbf{p}^{(1)} + \mathbf{p}^{(2)} = 0, \quad (49)$$

$$\mathbf{p}^{(1)} = -\mathbf{p}^{(2)} = \mathbf{p} = -i\nabla, \quad (50)$$

and

$$\mathbf{r} = \mathbf{r}^{(1)} - \mathbf{r}^{(2)}. \quad (51)$$

Since $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$ commute we can represent them by left and right multiplication of Φ by Dirac matrices so that

$$\gamma_\mu^{(1)}\Phi = \gamma_\mu\Phi, \quad \gamma_\mu^{(2)}\Phi = \Phi\gamma_\mu. \quad (52)$$

We now make a unitary non-local transformation similar to the Foldy-Wouthuysen transformation by defining a new two-body wave function

$$\tilde{\Phi} = W\Phi = W_1W_2\Phi, \quad (53)$$

where

$$W_i = \left[\frac{m_i + S/2 + i\boldsymbol{\gamma}^{(i)}\cdot\mathbf{p}^{(i)}}{\sqrt{(m_i + S/2)^2 + \mathbf{p}^{(i)}\cdot\mathbf{p}^{(i)}}} \right]^{1/2} \quad (54)$$

and $W_i W_i^\dagger = 1$ for $i = 1, 2$. Since \vec{x} does not commute with $\mathbf{p}^{(i)}$ the exact form of the transformed Hamiltonian \tilde{H} is very complicated. Instead we use an approximation for W by replacing S by an average value

$$\langle S \rangle = S(r_0) \quad (55)$$

where r_0 is the semiclassical radius of the Bohr orbit associated with the $q - \bar{q}$ potential. In this simplified version W depends only on differential operators, so that we can calculate \tilde{H} by expanding W in the powers of the momenta. \tilde{H} will also involve a transformed vector potential $\tilde{V}(r)$,

$$\tilde{H} = W H W^\dagger, \quad \tilde{V}(r) = W V(r) W^\dagger. \quad (56)$$

In the approximation that $\tilde{S}(r)$ is spin-independent, we have

$$\tilde{S}(r) = W S(r) W^\dagger \approx S(r) = S(r_0) + ((S(r) - S(r_0))), \quad (57)$$

$$\tilde{V}(r) \approx V(r) + K \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{4m_1 m_2} \nabla^2 V(r), \quad (58)$$

so that $\tilde{V}(r)$ differs from $V(r)$ by a spin-dependent Breit term.

The new two-body equation now reads

$$\tilde{H} \tilde{\Phi} = \tilde{V}(r) \tilde{\Phi} + \sqrt{[m_1 + \frac{1}{2} S(r)]^2 + \mathbf{p}_{(1)}^2} \gamma_4 \tilde{\Phi} + \sqrt{[m_2 + \frac{1}{2} S(r)]^2 + \mathbf{p}_{(2)}^2} \tilde{\Phi} \gamma_4. \quad (59)$$

This equation is invariant under the spin transformation

$$\tilde{\Phi} \rightarrow e^{\frac{1}{2} i \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_1} \Phi e^{\frac{-1}{2} i \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_2} \quad (60)$$

since

$$[\boldsymbol{\sigma}, \gamma_4] = 0, \quad (61)$$

provided the Breit term is neglected. The potential being flavor-independent, flavor independence is only broken by the flavor-dependent mass differences $m_1 - m_2$. This leads to an approximate $SU(6) \times SU(6)$ symmetry of \tilde{H} broken by mass differences and spin dependent Breit terms. Note that the original Hamiltonian H does not display this invariance. The invariance of \tilde{H} does not conflict with relativity since it is obtained by a non-local transformation.

Iteration of Eq.(59) gives the "Krolikowski" type second order equation

$$-\nabla^2 \tilde{\Phi} = \frac{1}{4} \left[1 - \left(\frac{m_1 - m_2}{E - \tilde{V}(r)} \right)^2 \right] [(E - \tilde{V}(r))^2 - (m_1 + m_2 + S(r))^2] \tilde{\Phi} \quad (62)$$

which was successfully used by Lichtenberg and his collaborators^[8] to calculate meson masses in the approximation $\tilde{V} = V$.

For a $q - D$ system $\bar{q}(2)$ is replaced by a wave function with four spin components associated with $s = 1$ and $s = 0$. In addition there are flavor quantum numbers. We write

$$\tilde{V}(r) = V(r) + K \frac{\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}}{m_1 m_2} \nabla^2 V(r), \quad (63)$$

with m_1 and m_2 being respectively the quark and diquark masses, $S^{(1)}$ the $s = 1/2$ Pauli matrices and $S^{(2)}$ being either zero or the 3×3 $s=1$ matrices. Otherwise the two-body equation is the same as for the meson case. If $m_1 - m_2$ and the spin-dependent terms are neglected the two-body equation for the meson-baryon system in which the wave function is

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Phi}_{q\bar{q}} & \tilde{\Psi}_{qD} \\ \tilde{\Psi}_{\bar{D}\bar{q}} & \tilde{\Phi}_{\bar{D}D} \end{pmatrix} \quad (64)$$

becomes invariant under the transformation

$$\tilde{\Psi} \rightarrow U \tilde{\Psi} V^\dagger \quad (65)$$

where U and V are elements of the Miyazawa supergroup $SU(6/21)$. In a relativistic treatment we can represent the diquark for each flavor by a 16-component wave function satisfying a linear Kemmer type equation. 10 components are for $s = 1$ ($F_{\mu\nu}$ and A_λ) and 5 are for $s = 0$ (ϕ_μ and ϕ). Adding 4 Dirac components for the quark with $s = 1$, we find that for each flavor state the $q - \bar{D}$ system has 20 covariant components representing 6 physical spin states $s = \frac{1}{2}, 0$ and 1.

The matrix wave function $\tilde{\Psi}$ obtained from the covariant wave function after a non-local transformation has the physical states separated in a non-covariant way expressible in terms of 27×27 matrix. It satisfies a wave equation of the form

$$H\tilde{\Psi} = i\frac{\partial\tilde{\Psi}}{\partial t} = \{K, \tilde{\Psi}\} + \mathbf{S}\tilde{\Psi}\mathbf{S} \quad (66)$$

where

$$K = \begin{pmatrix} K_q I^{(6)} & 0 \\ 0 & K_D I^{(21)} \end{pmatrix} \quad (67)$$

with $I^{(n)}$ denoting the $n \times n$ unit matrix and

$$K_q = -\frac{2}{3}\frac{\alpha_s}{r} + \gamma_4[(m_q + \frac{1}{2}S)^2 - \nabla^2]^{\frac{1}{2}}, \quad (68)$$

$$K_D = -\frac{2}{3}\frac{\alpha_s}{r} + \beta_4[(m_q + \frac{1}{2}S)^2 - \nabla^2]^{\frac{1}{2}}, \quad (69)$$

$$\mathbf{S} = K^{\frac{1}{2}} \begin{pmatrix} m_q^{-1} \mathbf{S}_q I^{(6)} & 0 \\ 0 & m_D^{-1} \mathbf{S}_D I^{(21)} \end{pmatrix} \quad (70)$$

where β_4 is the fourth Petiau-Kemmer matrix for $s = 0$ or 1 can be represented by

$$\beta_4 = \gamma_4^{(2)} + \gamma_4^{(3)} \quad (71)$$

referring to the two quark constituents of the diquark. \mathbf{S}_q and \mathbf{S}_D are respectively the quark and diquark spin operators, m_q and m_D are their respective masses. The second term in Eq.(66) gives the spin dependent Breit term and breaks the approximate $SU(6/21)$ symmetry generated by

$$\delta\tilde{\Psi} = [X, \tilde{\Psi}], \quad (72)$$

$$X = \begin{pmatrix} M & B \\ \bar{B} & N \end{pmatrix} \quad (73)$$

where B and \bar{B} are Grassmann valued, M and N are antihermitian 6×6 and 21×21 matrices respectively. S is the confining scalar potential. We note that

$$-\nabla^2 = \mathbf{p}^2 = p_r^2 + \frac{\ell(\ell+1)}{r^2}, \quad (74)$$

where p_r is canonically conjugate to r .

The ground state energy eigenvalue E of the Hamiltonian can be estimated by using the Heisenberg uncertainty principle. This leads to the replacement of r by Δr and p_r by

$$\Delta p_r = \frac{1}{2}(\Delta r)^{-1}, \quad (h = 1). \quad (75)$$

Then E as a function of Δr is minimized for the value of r_0 of Δr . The r_0 corresponds to the Bohr radius for the bound state. The confining energy associated with this Bohr radius is obtained from the linear confining potential

$$S(r) = br. \quad (76)$$

so that the effective masses of the constituents become

$$M_1 = m_1 + \frac{1}{2}S_0, \quad M_2 = m_2 + \frac{1}{2}S_0 \quad (77)$$

$$S_0 = br_0. \quad (78)$$

For a meson m_1 and m_2 are the current quark masses while M_1 and M_2 can be interpreted as the constituent quark masses. Note that even in the case of vanishing quark masses associated with perfect chiral symmetry, confinement results in non zero constituent masses that spontaneously break the $SU(2) \times SU(2)$ symmetry of the u, d quarks.

Let us illustrate this method on the simplified spin free Hamiltonian involving only the scalar potential. In the center of mass system, $\mathbf{p}^{(1)} + \mathbf{p}^{(2)} = 0$, or $\mathbf{p}^{(1)} = -\mathbf{p}^{(2)} = \mathbf{p}$. The semi-relativistic Hamiltonian of the system is then given by

$$E_{12} \Phi = \sum_{i=1}^2 \sqrt{(m_i + \frac{1}{2}br)^2 + \mathbf{p}^2} \Phi. \quad (79)$$

Taking $m_1 = m_2 = m$ for the quark-antiquark system, we have

$$E_{12} \Phi = 2\sqrt{(m + \frac{1}{2}br)^2 + p_r^2 + \frac{\ell(\ell+1)}{r^2}} \Phi, \quad (80)$$

where we have written the momentum part in spherical coordinates.

Putting

$$b = \mu^2, \quad \rho = \mu \quad r, \quad (81)$$

for the $q - \bar{q}$ system we find E_{12} by minimizing the function

$$E_{q\bar{q}} = 2\sqrt{(m + \frac{1}{2}\mu\rho)^2 + \frac{\mu^2}{\rho^2}(\ell + \frac{1}{2})^2}. \quad (82)$$

For u and d quarks, m is small and can be neglected so that

$$E^2 = \mu^2[\rho^2 + \rho^{-2}(2\ell + 1)^2] \quad (83)$$

which has a minimum for

$$\rho^2 = \rho_0^2 = 2\ell + 1, \quad (84)$$

giving

$$E_{min}^2 = E^2(\rho_0) = 4\mu^2(\ell + \frac{1}{2}). \quad (85)$$

Thus, we obtain a linear Regge trajectory with

$$\alpha' = \frac{1}{4}\mu^{-2} = \frac{b}{4}. \quad (86)$$

Also

$$\mathbf{J} = \boldsymbol{\ell} + \mathbf{S} \quad (87)$$

where \mathbf{S} arises from the quark spins. Experimentally

$$\alpha' = 0.88(GeV)^{-2} \quad (88)$$

for mesons giving the value 0.54 GeV for μ . A more accurate calculation^{[11],[12]} gives

$$\alpha' = (2\pi\mu^2)^{-1}, \quad \mu \sim 0.43 GeV. \quad (89)$$

The constituent quark mass can be defined in two ways

$$M_c(\ell) = \frac{1}{2}E_{min} = \mu\sqrt{\ell + \frac{1}{2}}, \quad (90)$$

or

$$m_c'(\ell) = S_0 = \frac{1}{2}\mu\rho_0 = \frac{\mu}{\sqrt{2}}\sqrt{\ell + \frac{1}{2}}. \quad (91)$$

The first definition gives for $\ell = 0$,

$$M_c = 0.31 \text{ GeV} \quad \text{for} \quad \mu = 0.43 \quad (92)$$

in the case of u and d quarks.

When the Coulomb like terms are introduced in the simplified Hamiltonian Eq.(80) with negligible quark masses one obtains

$$E = \frac{\mu}{\rho}[-\bar{\alpha} + \sqrt{\rho^4 + (2\ell + 1)^2}] \quad (93)$$

with

$$\bar{\alpha} = \frac{4}{3}\alpha_s \quad \text{for} \quad (q\bar{q}), \quad \bar{\alpha} = \frac{2}{3}\alpha_s \quad \text{for} \quad (qq). \quad (94)$$

In the energy range around 1 GeV α_s is of order of unity. Estimates range from 0.3 to 3. Minimization of E gives

$$E_0 = \mu u_0^{-\frac{1}{4}}(-\bar{\alpha} + \sqrt{u_0 + (2\ell + 1)^2}) \quad (95)$$

where

$$u_0(\epsilon) = \rho_0^4 = (2\ell + 1)^2(1 + \frac{1}{2}\beta^2 + \epsilon\sqrt{2}\beta\sqrt{\ell + \frac{1}{8}}\beta^2), \quad (96)$$

$$\epsilon = \pm 1, \quad \beta = \frac{\bar{\alpha}}{(2\ell + 1)}. \quad (97)$$

The minimum E_0 is obtained for $\epsilon = -1$, giving to second order in β :

$$E_0 = \mu\sqrt{2(2\ell + 1)}(1 - \frac{\beta}{\sqrt{2}} - 3\frac{\beta^2}{8}). \quad (98)$$

Linear Regge trajectories are obtained if β^2 is negligible. Then for mesons

$$E_0^2 = 4\mu^2\ell + 2\mu^2(1 - \sqrt{2}\bar{\alpha}). \quad (99)$$

The β^2 is negligible for small ℓ only if we take the lowest estimate for α_s , giving 0.4 for $\bar{\alpha}$ in the $q\bar{q}$ case. For mesons with u , d constituents, incorporating their spins through the Breit term we obtain approximately

$$m_\rho \simeq m_\omega = E_0 + \frac{c}{4}, \quad m_\pi = E_0 - \frac{3c}{4}, \quad (100)$$

$$c = K \frac{\Delta V}{M_q^2} \quad (101)$$

where M_q is the constituent quark mass. This gives

$$E_0 = \frac{(3m_\rho + m_\pi)}{4} = 0.61 GeV. \quad (102)$$

The Regge slope being of the order of $1 GeV$ an average meson mass of the same order is obtained from Eq.(98) in the linear trajectory approximation. To this approximation $\bar{\alpha}$ should be treated like a parameter rather than be placed by its value derived from QCD under varying assumptions. Using the value shown in Eq.(89) for μ , one gets a better fit to the meson masses by taking $\alpha_s \sim 0.2$.

Turning now to baryon masses, we must first estimate the diquark mass. We have for the qq system

$$M_D = \mu(\sqrt{2} - \frac{2}{3}\alpha_s), \quad (103)$$

that is slightly higher than the average meson mass

$$\tilde{m} = \mu(\sqrt{2} - \frac{4}{3}\alpha_s) \quad (104)$$

Here we note that E is not very sensitive to the precise value of the QCD running coupling constant in the GeV range. Taking $\alpha_s \sim 0.3$ changes E^{qq} from 0.55 to 0.56 GeV .

Note that Eq.(103) gives $M_D = 0.55 GeV$. For excited $q - \bar{q}$ and $q - D$ systems if the rotational excitation energy is large compared with μ , then both the m_D and the Coulomb term $-\frac{4}{3}\frac{\alpha_s}{r}$ (same for $q - D$ and $q - \bar{q}$ systems) can be neglected.

Thus, for both $(q - D)$ [excited baryon] and $q - \bar{q}$ [excited meson] systems we have Eq.(85), namely

$$(E^{q-D})^2 \sim (E^{q-\bar{q}})^2 \sim 4\mu^2\ell + 2\mu^2 \quad (105)$$

giving again Eq.(86), i.e.

$$(\alpha')_{q-D} = (\alpha')_{q-\bar{q}} \cong \frac{1}{4\mu^2} \quad \text{or} \quad \left(\frac{1}{2\pi\mu^2}\right) \quad (106)$$

as an explanation of hadronic supersymmetry in the nucleon and meson Regge spectra. We also have, extrapolating to small ℓ :

$$\Delta(M^2)^{q-D} = \Delta(m^2)^{q-\bar{q}} = 4\mu^2 \Delta\ell = \frac{1}{\alpha'} \Delta\ell. \quad (107)$$

For $\Delta\ell = 1$ we find

$$m_\Delta^2 - m_N^2 = m_\rho^2 - m_\pi^2. \quad (108)$$

This relationship is same as the one proved by Miyazawa through the assumption that $U(6/21)$ symmetry is broken by an operator that behaves like $s = 0$, $I = 0$ member of 35×35 representations of $SU(6)$, which is true to 5%. It corresponds to a confined quark approximation with $\alpha_s = 0$.

The potential model gives a more accurate symmetry breaking ($\alpha_s \sim 0.2$):

$$\frac{9}{8}(m_\rho^2 - m_\pi^2) = m_\Delta^2 - m_N^2 \quad (109)$$

with an accuracy of 1%, in a remarkable agreement with experiments.

This mass squared formula arises from the second order iteration of the $q - D$, $q - \bar{q}$ Dirac equation. The factor $\frac{9}{8}$ comes from

$$\frac{1}{2}\left(\frac{4}{3}\alpha_s\right)^2 = \frac{8}{9}\alpha_s^2. \quad (110)$$

At this point it is more instructive to derive a first order mass formula.

Since the constituent quark mass M_q is given by Eq.(90) ($\ell = 0$), we have

$$M_q = \frac{\mu}{\sqrt{2}}, \quad (111)$$

so that

$$\bar{m} = 2M_q\left(1 - \frac{\sqrt{2}}{3}\alpha_s\right) \simeq 1.9M_q. \quad (112)$$

When the baryon is regarded as a $q - D$ system, each constituent gains an effective mass $\frac{1}{2}\mu\rho_0$ which was approximately the effective mass of the quark in the meson. Hence, the effective masses of q and D in the baryon are

$$m'_q \simeq M_q, \quad m'_D = M_D + M_q \simeq 3M_q. \quad (113)$$

The spin splittings for the nucleon N and the Δ are given by the Breit term

$$\Delta M = K\Delta V \frac{\mathbf{S}_q \cdot \mathbf{S}_D}{m'_q m'_D}. \quad (114)$$

For the nucleon with spin $\frac{1}{2}$ the term $\mathbf{S}_q \cdot \mathbf{S}_D$ gives -1 while it has the value $\frac{1}{2}$ for Δ with spin $\frac{3}{2}$. Using the same K for mesons and baryons which are both considered to be a bound state of a color triplet with a color antitriplet we can relate the baryon splitting ΔM to the meson splitting Δm for which $\mathbf{S}_q \cdot \mathbf{S}_{\bar{q}}$ takes the values $\frac{1}{4}$ and $\frac{-3}{4}$. Hence we find

$$\Delta M = M_\Delta - M_N = \frac{3}{2} \frac{K\Delta V}{m'_q m'_D} = \frac{3}{2} \frac{K\Delta V}{3M_q^2} \quad (115)$$

$$\Delta m = \frac{K\Delta V}{M_q^2} \quad (116)$$

which leads to a linear mass formula

$$\Delta M = \frac{1}{2} \Delta m \quad (117)$$

which is well satisfied and has been verified before using the three quark constituents for the baryon.

The formation of diquarks which behave like antiquarks as far as QCD is concerned is crucial to hadronic supersymmetry and to quark dynamics for excited hadrons. The splittings in the mass spectrum are well understood on the basis of spin-dependent terms derived from QCD. This approach to hadronic physics has led to many in depth investigations recently. Examples are the work of Martin^[24], Yamagishi^[25] and others, and more recently Klempt^[42], and Lichtenberg and collaborators^[8] and references contained in their review paper^[9].

Let us now consider a quark-antiquark system in the approximation that the potential is only a scalar. In the center of mass system, $\mathbf{p}^{(1)} + \mathbf{p}^{(2)} = 0$, or $\mathbf{p}^{(1)} = -\mathbf{p}^{(2)} = \mathbf{p}$. The semi-relativistic Hamiltonian of the system is then given by

$$H = \sum_{i=1}^2 \sqrt{(m_i + \frac{1}{2}br)^2 + \mathbf{p}^2}. \quad (118)$$

Taking $m_1 = m_2 = m$ for the quark-antiquark system, we have

$$H = 2\sqrt{(m + \frac{1}{2}br)^2 + p_r^2 + \frac{\ell(\ell+1)}{r^2}}, \quad (119)$$

where we have written the momentum part in spherical coordinates. For u and d quarks, m is small and can be neglected so that

$$H^2 = 4[\frac{1}{4}b^2r^2 + p_r^2 + \frac{\ell(\ell+1)}{r^2}]. \quad (120)$$

Similar equations for a $q-\bar{q}$ meson system were already proposed and solved numerically^[8] or exactly^[26] by several authors. The eigenfunction Ψ for H^2 has eigenvalue E^2 such that

$$4[\frac{1}{4}b^2r^2 - \frac{1}{r}\frac{d^2}{dr^2}r + \frac{\ell(\ell+1)}{r^2}]\Psi = E^2\Psi. \quad (121)$$

The differential equation can be solved exactly, and the normalized eigenfunction is found to be

$$\begin{aligned} \Psi(r, \theta, \phi) = & \left[\frac{2(\frac{b}{2})^{2\ell+3}\Gamma(\ell + \frac{3}{2} + n_r - 1)}{(n_r - 1)!\Gamma^2(\ell + \frac{3}{2})} \right]^{1/2} r^\ell \exp[-\frac{b}{4}r^2] \times \\ & F(-n_r + 1, \ell + \frac{3}{2}, \frac{b}{2}r^2) Y_\ell^m(\theta, \phi), \end{aligned} \quad (122)$$

where $F(-n_r + 1, \ell + \frac{3}{2}, \frac{b}{2}r^2)$ is the confluent hypergeometric function, and $n_r = 1, 2, 3, \dots$, is the radial quantum number. The eigenvalue is given by

$$E^2 = 4b[2(n_r - 1) + \ell + \frac{3}{2}], \quad (123)$$

therefore we obtain linear Regge trajectories of slope $1/4b$ when we make plots of ℓ versus M^2 . The case of $n_r = 1$ corresponds to the leading Regge trajectory, and cases of $n_r = 2, 3, \dots$, correspond to the parallel daughter trajectories.

Iachello and his collaborators^[27] have obtained a similar mass formula based on algebraic methods. Starting from a spectrum generating algebra G , we can write a chain of subalgebras

$$G \supset G' \supset G'' \supset \cdots. \quad (124)$$

They proposed that the Hamiltonian can be expanded in terms of invariants of the chain of subalgebras

$$H = \alpha C(G) + \beta C(G') + \gamma C(G'') + \cdots, \quad (125)$$

where $C(G)$ denotes one of the invariants of G . In a two-body problem, the spectrum generating algebra is $U(4)$, and in the total symmetric representation, one of the two chains of subalgebras is $U(3)$ followed by $SO(3)$ with respective Casimir invariants $n(n+2)$ and $\ell(\ell+1)$, where n corresponds to the vibrational mode. In order to derive the relativistic mass formula like Eq.(123), one has to write a formula for the square of the mass, rather than the mass itself. Furthermore, without violating the dynamic symmetry, the Hamiltonian can be written in terms of non-linear functions of the Casimir invariants. Therefore, we can have

$$\begin{aligned} M^2 &= \alpha + \beta \sqrt{n(n+2) + 1} + \gamma \sqrt{\ell(\ell+1) + 1/4} \\ &= \alpha' + \beta n + \gamma \ell. \end{aligned} \quad (126)$$

Comparing this mass formula with Eq.(123), we find that the relativistic quark model suggests a relation of $\beta = 2\gamma$ in Eq.(126). However, the above algebraic method treats β and γ as two independent parameters, and so the extra degree of freedom allows Eq.(126) a better fit to the experimental data. The fitted parameters for mesons are found to be: $\beta \approx 1.5 \text{ (GeV)}^2$ and $\gamma \approx 1.1 \text{ (GeV)}^2$.

The inclusion of the Coulomb-like term causes deviations from the linear trajectories at low energies and changes the relation between the parameters β and γ . It is important to carry out a calculation based on a better approximation to see if the empirical values of these parameters are compatible with our model.

A meson is made up of a quark and an antiquark, while a baryon is made up of three quarks. The complicated three-body problem in a baryon can be simplified to a two-body one if we assume two of the three constituent quarks form a diquark system. We have recently^{[10],[13]} discussed a possible hadronic supersymmetry of baryons and mesons suggested by the symmetry between a quark-diquark system and a quark-antiquark system.

In the color group $SU(3)$ of QCD, a quark is in the $(\mathbf{3})$ representation, while an antiquark is in the $(\overline{\mathbf{3}})$ representation. Since $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}$, a diquark can be in $(\mathbf{6})$ or $(\overline{\mathbf{3}})$. The color force due to one gluon exchange is repulsive in the $(\mathbf{6})$ representation, attractive in the $(\overline{\mathbf{3}})$ representation, with the attractive force between two quarks being half in strength as that of between a quark and an antiquark. Therefore, a diquark is energetically favored to be in the $(\overline{\mathbf{3}})$. From the representations of the color group $SU(3)$, QCD suggests a symmetry between an antiquark \overline{q} and a diquark D , both of $(\overline{\mathbf{3}})$ representation, and similarly for a quark q and an antidiquark \overline{D} in the $(\mathbf{3})$ representation.

In the spin-flavor $SU(6)$ symmetry, q and D lie respectively in the $(\mathbf{6})$ and $(\overline{\mathbf{21}})$ representations of $SU(6)$. One can put a quark (fermion) and an antidiquark (boson) into a supersymmetric multiplet

$$\psi = \begin{pmatrix} q \\ D \end{pmatrix} \quad (127)$$

The effective supersymmetry can be described by the graded algebra $SU(6/21)$ with 6 fermionic and 21 bosonic components. We obtain a 27×27 hadron multiplet by forming the matrix

$$\psi \overline{\psi} = \begin{pmatrix} q\overline{q} & qD \\ \overline{D}\overline{q} & \overline{D}D \end{pmatrix}. \quad (128)$$

This scheme predicts the existence of exotic mesons $\overline{D}D$, as well as mesons $(q\overline{q})$, baryons (qD) and antibaryons $(\overline{D}\overline{q})$.

The supersymmetry produces naturally the result that baryons and mesons have the same slope of their Regge trajectories. This prediction agrees with experiment at high angular momenta. Experimentally, the slopes of the Regge trajectories for mesons and baryons are both approximately $0.9 (\text{GeV})^{-2}$. Therefore, the scalar potential $S(r) = br$ has the fitted parameter $b \approx 0.3 (\text{GeV})^2$.

To explain the meson-baryon symmetry phenomenon, note that at high angular momenta, a baryon resembles a string-like object with a quark at one end and a diquark at the other. The gluon flux tube contracts to a very small cross-sectional area, and hence the two quarks forming the diquark must be very close together to make the diquark look like a single point particle. Neglecting the difference between the diquark and antiquark masses, the baryon system becomes a meson-like system. Both D and \overline{q} are in the $(\overline{\mathbf{3}})$

color representation, so that the q - D potential is the same as the q - \bar{q} potential provided we neglect the spin dependence of the color force. The calculation of the mass spectrum of mesons discussed earlier just applies to baryons as well, so the Regge trajectories of mesons and baryons are parallel. This remarkable manifestation of hadronic supersymmetry had found no simple explanation based on QCD until the quark-diquark model was worked out.

The observed hadronic $SU(6/21)$ supersymmetry is not exact. The breaking of this supersymmetry has two origins. First one is the q and D mass differences as well as mass differences among quarks. This leads to different intercepts for parallel Regge trajectories. The second breaking comes from the contribution to the potential from one gluon exchange. This potential is a 4-vector and is spin dependent. Since the quark and antiquark have spin $S = 1/2$ and the diquark has $S = 0$ or $S = 1$, the spin dependent part of the q - D potential is different from that of q - \bar{q} , causing supersymmetry breaking. Another effect is the deviation of the Regge trajectories from linearity for low angular momenta, since the potential is no longer proportional to the distance, and quark masses can no longer be neglected.

To see the symmetry breaking effect, we consider a hadron that approximates a two-body system. The quark, antiquark, and diquark will acquire their effective masses under the influence of the effective potentials. There are also spin-dependent interactions among the quarks. Based on a semi-relativistic Hamiltonian for a quark-diquark system interacting with the same potentials $\tilde{S}(r)$ and the $\tilde{V}(r)$ the mass of a hadron will take the approximate form

$$m_{12} = m_1 + m_2 + K \frac{\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}}{m_1 m_2}, \quad (129)$$

where m_i and $\mathbf{S}^{(i)}$ ($i = 1, 2$) are respectively the constituent mass and the spin of a quark or a diquark. The spin-dependent Breit term will split the masses of hadrons of different spin values.

If we assume

$$m_q = m_{\bar{q}} = m \quad (130)$$

where m is the constituent mass of u or d quarks, and denote the mass of diquark as m_D , then this approximation gives

$$m_\pi = (m_{q\bar{q}})_{S=0} = 2m - K \frac{3}{4m^2}, \quad (131)$$

$$m_\rho = (m_{q\bar{q}})_{S=1} = 2m + K \frac{1}{4m^2}, \quad (132)$$

$$m_\Delta = (m_{qD})_{S=3/2} = m + m_D + K \frac{1}{2mm_D}, \quad (133)$$

$$m_N = (m_{qD})_{S=1/2} = m + m_D - K \frac{1}{mm_D}. \quad (134)$$

Eliminating m , m_D and K , we obtain a mass relation

$$\frac{8}{3} \cdot \frac{2m_\Delta + m_N}{3m_\rho + m_\pi} = 1 + \frac{3}{2} \cdot \frac{m_\rho - m_\pi}{m_\Delta - m_N}, \quad (135)$$

which agrees with experiment to 13%.

The supersymmetry based on the supergroup $U(6/21)$ acts on a quark and antiquark situated at the same point x_1 . At the point x_2 we can consider the action of a supergroup with the same parameters, or one with different parameters. In the first case we have a global symmetry. In the second case, if we only deal with bilocal fields, the symmetry will be represented by $U(6/21) \times U(6/21)$, doubling the Miyazawa supergroup. On the other hand, if any number of points are considered, with different parameters attached to each point, we are led to introduce a local supersymmetry $U(6/21)$ to which we should add the local color group $SU(3)^c$. Since it is not a fundamental symmetry, we shall not deal with the local Miyazawa group here. However, the double Miyazawa supergroup is useful for bilocal fields since the decomposition of the adjoint representation of the 728-dimensional Miyazawa group with respect to $SU(6) \times SU(21)$ gives

$$728 = (35, 1) + (1, 440) + (6, 21) + (\bar{6}, \bar{21}) + (1, 1) \quad (136)$$

A further decomposition of the double Miyazawa supergroup into its field with respect to its c.m. coordinates leads to the decomposition of the 126-dimensional cosets $(6, 21)$ and $(21, 6)$ into $56^+ + 70^-$ of the diagonal $SU(6)$.

We would have a much tighter and more elegant scheme if we could perform such a decomposition from the start and be able to identify the $(1, 21)$ part of the fundamental representation of $U(6/21)$ with the 21-dimensional representation of the $SU(6)$ subgroup, which means going beyond the Miyazawa supersymmetry to a smaller supergroup having $SU(6)$ as a subgroup. Next, we describe the development of a bilocal treatment, and a new minimal scheme where the bilocal treatment gets carried over unchanged into it.

Bilocal approximation to hadronic structure and inclusion of color.

Low-lying baryons occur in the symmetric 56 representation^[29] of $SU(6)$, whereas the Pauli principle would have led to the antisymmetrical 20 representation. This was a crucial fact for the introduction of color degree of freedom^[30] based on $SU(3)^c$. Since the quark field transforms like a color triplet and the diquark like a color antitriplet under $SU(3)^c$, the color degrees of freedom of the constituents must be included correctly in order to obtain a correct representation of the q-D system. Hadronic states must be color singlets. These are represented by bilocal operators $O(\mathbf{r}_1, \mathbf{r}_2)$ in the bilocal approximation^[31] that gives $\bar{q}(1)q(2)$ for mesons and $D(1)q(2)$ for baryons. Here $\bar{q}(1)$ represents the antiquark situated at \mathbf{r}_1 , $q(2)$ the quark situated at \mathbf{r}_2 , and $D(1) = q(1)q(1)$ the diquark situated at \mathbf{r}_1 . If we denote the c.m. and the relative coordinates of the constituents by \mathbf{R} and \mathbf{r} , where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and

$$\mathbf{R} = \frac{(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)}{(m_1 + m_2)} \quad (137)$$

with m_1 and m_2 being their masses, we can then write $O(\mathbf{R}, \mathbf{r})$ for the operator that creates hadrons out of the vacuum. The matrix element of this operator between the vacuum and the hadronic state h will be of the form

$$\langle h | O(\mathbf{R}, \mathbf{r}) | 0 \rangle = \chi(\mathbf{R}) \psi(\mathbf{r}) \quad (138)$$

where $\chi(\mathbf{R})$ is the free wave function of the hadron as a function of the c.m. coordinate and $\psi(\mathbf{r})$ is the bound-state solution of the $U(6/21)$ invariant Hamiltonian describing the $q - \bar{q}$ mesons, $q - D$ baryons, $\bar{q} - \bar{D}$ antibaryons and $D - \bar{D}$ exotic mesons, given by

$$i\partial_t \psi_{\alpha\beta} = \left[\sqrt{(m_\alpha + \frac{1}{2}V_s)^2 + \mathbf{p}^2} + \sqrt{(m_\beta + \frac{1}{2}V_s)^2 + \mathbf{p}^2} - \frac{4}{3} \frac{\alpha_s}{r} + k \frac{\mathbf{s}_\alpha \cdot \mathbf{s}_\beta}{m_\alpha m_\beta} \right] \psi_{\alpha\beta} \quad (139)$$

Here $\mathbf{p} = -i\nabla$ in the c.m. system and m and s denote the masses and spins of the constituents, α_s the strong-coupling constant, $V_s = br$ is the scalar potential with r being the distance between the constituents in the bilocal object, and $k = |\psi(0)|^2$.

The operator product expansion^[32] will give a singular part depending only on \mathbf{r} and proportional to the propagator of the field binding the two constituents. There will be a finite number of singular coefficients $c_n(\mathbf{r})$ depending on the dimensionality of the constituent fields. For example, for a meson, the singular term is proportional to the propagator of the gluon field binding the two constituents. Once we subtract the singular part, the remaining part $\tilde{O}(\mathbf{R}, \mathbf{r})$ is analytic in \mathbf{r} and thus we can write

$$\tilde{O}(\mathbf{R}, \mathbf{r}) = O_0(\mathbf{R}) + \mathbf{r} \cdot \mathbf{O}_1(\mathbf{R}) + o(r^2). \quad (140)$$

Now $O_0(\mathbf{R})$ creates a hadron at its c.m. point \mathbf{R} equivalent to a $\ell = 0$, s-state of the two constituents. For a baryon this is a state associated with q and $D \sim qq$ at the same point \mathbf{R} , hence it is essentially a 3-quark state when the three quarks are at a common location. The $\mathbf{O}_1(\mathbf{R})$ can create three $\ell = 1$ states with opposite parity to the state created by $O_0(\mathbf{R})$. Hence, if we denote the nonsingular parts of $\bar{q}(1)q(2)$ and $D(1)q(2)$ by $[\bar{q}(1)q(2)]$ and $[D(1)q(2)]$, respectively, we have

$$[\bar{q}(1)q(2)]|0\rangle = |M(\mathbf{R})\rangle + \mathbf{r} \cdot |\mathbf{M}'(\mathbf{R})\rangle + o(r^2) \quad (141)$$

$$[D(1)q(2)]|0\rangle = |B(\mathbf{R})\rangle + \mathbf{r} \cdot |\mathbf{B}'(\mathbf{R})\rangle + o(r^2) \quad (142)$$

and similarly for the exotic meson states $D(1)\bar{D}(2)$.

Here M belongs to the $(35 + 1)$ -dimensional representation of $SU(6)$ corresponding to an $\ell = 0$ bound state of the quark and the antiquark. The $\mathbf{M}'(\mathbf{R})$ is an orbital excitation ($\ell = 1$) of opposite parity, which are in the $(35 + 1, 3)$ representation of the group $SU(6) \times O(3)$, $O(3)$ being associated with the relative angular momentum of the constituents. The \mathbf{M}' states contain mesons like B , \mathbf{A}_1 , \mathbf{A}_2 and scalar particles. On the whole, the $\ell = 0$ and $\ell = 1$ part $\bar{q}(1)q(2)$ contain $4 \times (35 + 1) = 144$ meson states.

Switching to the baryon states, the requirement of antisymmetry in color and symmetry in spin-flavor indices gives the $(56)^+$ representation for $B(\mathbf{R})$. The $\ell = 1$ multiplets have negative parity and have mixed spin-flavor symmetry. They belong to the representation $(70^-, 3)$ of $SU(6) \times O(3)$ and are represented by the states $|\mathbf{B}'(\mathbf{R})\rangle$ which are 210 in number. On the whole, these 266 states account for all the observed low-lying baryon states obtained from $56 + 3 \times 70 = 266$. A similar analysis can be carried out for the exotic meson states $D(1)\bar{D}(2)$, where the diquark and the antidiquark can be bound in an $\ell = 0$ or $\ell = 1$ state with opposite parities.

Color Algebra and Octonions

The exact, unbroken color group $SU(3)^c$ is the backbone of the strong interaction. It is worthwhile to understand its role in the diquark picture more clearly.

Two of the colored quarks in the baryon combine into an anti-triplet $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + (\mathbf{6})$, $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + (\mathbf{8})$. The $(\mathbf{6})$ partner of the diquark and the $(\mathbf{8})$ partner of the nucleon do not exist. In hadron dynamics the only color combinations to consider are $\mathbf{3} \times \mathbf{3} \rightarrow \bar{\mathbf{3}}$ and $\bar{\mathbf{3}} \times \mathbf{3} \rightarrow \mathbf{1}$. These relations imply the existence of split octonion units through a representation of the Grassmann algebra $\{u_i, u_j\} = 0$, $i = 1, 2, 3$. What is a bit strange is that operators u_i , unlike ordinary fermionic operators, are not associative. We also have $\frac{1}{2}[u_i, u_j] = \epsilon_{ijk} u_k^*$. The Jacobi identity does not hold since $[u_i, [u_j, u_k]] = -ie_7 \neq 0$, where e_7 , anticommute with u_i and u_i^* .

The behavior of various states under the color group are best seen if we use split octonion units defined by^[33]

$$u_0 = \frac{1}{2}(1 + ie_7), \quad u_0^* = \frac{1}{2}(1 - ie_7) \quad (143)$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1, 2, 3 \quad (144)$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2 . The imaginary octonion units e_α ($\alpha = 1, \dots, 7$) fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* are singlets, while u_j and u_j^* correspond, respectively, to the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

$$u_0^2 = u_0, \quad u_0 u_0^* = 0 \quad (145)$$

$$u_0 u_j = u_j u_0^* = u_j, \quad u_0^* u_j = u_j u_0 = 0 \quad (146)$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^* \quad (147)$$

$$u_i u_j^* = -\delta_{ij} u_0 \quad (148)$$

where ϵ_{ijk} is completely antisymmetric with $\epsilon_{ijk} = 1$ for $ijk = 123, 246, 435, 651, 572, 714, 367$. Here, one sees the virtue of octonion multiplication. If we consider the direct products

$$C : \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \quad (149)$$

$$G : \quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \quad (150)$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of $\mathbf{8}$ in $\mathbf{3} \otimes \bar{\mathbf{3}}$, while it gets rid of $\mathbf{6}$ in $\mathbf{3} \otimes \mathbf{3}$. Combining Eq.(147) and Wq.(148) we find

$$(u_i u_j) u_k = -\epsilon_{ijk} u_0^* \quad (151)$$

Thus the octonion product leaves only the color part in $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, so that it is a natural algebra for colored quarks.

For convenience we now produce the following multiplication table for the split octonion units:

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki} u_i^*$	$-\delta_{jk} u_0$
u_j^*	u_j^*	0	$-\delta_{jk} u_0^*$	$\epsilon_{jki} u_i$

It is worth noting that u_i and u_j^* behave like fermionic annihilation and creation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ij} \quad (152)$$

For more recent reviews on octonions and nonassociative algebras we refer to papers by Okubo^[34], and Baez^[35]. For generalized matrix representations of octonions we refer to Delbourgo, et al.^[36]

The quarks, being in the triplet representation of the color group $SU(3)^c$, are represented by the local fields $q_\alpha^i(x)$, where $i = 1, 2, 3$ is the color index and α the combined spin-flavor index. Antiquarks at point y are color antitriplets $q_\beta^i(y)$. Consider the two-body systems

$$C_{\alpha j}^{\beta i} = q_\alpha^i(x_1) \bar{q}_\beta^j(x_2) \quad (153)$$

$$G_{\alpha\beta}^{ij} = q_{\alpha}^i(x_1)q_{\beta}^j(x_2) \quad (154)$$

so that C is either a color singlet or color octet, while G is a color antitriplet or a color sextet. Now C contains meson states that are color singlets and hence observable. The octet $q-\bar{q}$ state is confined and not observed as a scattering state. In the case of two-body G states, the antitriplets are diquarks which, inside a hadron can be combined with another triplet quark to give observable, color singlet, three-quark baryon states. The color sextet part of G can only combine with a third quark to give unobservable color octet and color decuplet three-quark states. Hence the hadron dynamics is such that the **8** part of C and the **6** part of G are suppressed. This can best be achieved by the use of the above octonion algebra^[37]. The dynamical suppression of the octet and sextet states in Eq.(153) and Eq.(154) is, therefore, automatically achieved. The split octonion units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks we can define the "transverse" octonions (calling u_0 and u_0^* longitudinal units)

$$q_{\alpha} = u_i q_{\alpha}^i = \mathbf{u} \cdot \mathbf{q}_{\alpha}, \quad \bar{q}_{\beta} = u_i^{\dagger} \bar{q}_{\beta}^i = -\mathbf{u}^* \cdot \bar{\mathbf{q}}_{\beta} \quad (155)$$

We find

$$q_{\alpha}(1)\bar{q}_{\beta}(2) = u_0 \mathbf{q}_{\alpha}(1) \cdot \mathbf{q}_{\beta}(2) \quad (156)$$

$$\bar{q}_{\alpha}(1)q_{\beta}(2) = u_0^* \bar{\mathbf{q}}_{\alpha}(1) \cdot \mathbf{q}_{\beta}(2) \quad (157)$$

$$G_{\alpha\beta}(12) = q_{\alpha}(1)q_{\beta}(2) = \mathbf{u}^* \cdot \mathbf{q}_{\alpha}(1) \times \mathbf{q}_{\beta}(2) \quad (158)$$

$$G_{\beta\alpha}(21) = q_{\beta}(2)q_{\alpha}(1) = \mathbf{u}^* \cdot \mathbf{q}_{\beta}(2) \times \mathbf{q}_{\alpha}(1) \quad (159)$$

Because of the anticommutativity of the quark fields, we have

$$G_{\alpha\beta}(12) = G_{\beta\alpha}(21) = \frac{1}{2}\{q_{\alpha}(1), q_{\beta}(2)\} \quad (160)$$

If the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center-of-mass location x

$$x = \frac{1}{2}(x_1 + x_2) \quad (161)$$

when x_2 tends to x_1 we can replace the argument by x , and we obtain

$$D_{\alpha\beta}(x) = D_{\beta\alpha}(x) \quad (162)$$

so that the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter is taken to be $SU(6)$, then $D_{\alpha\beta}(x)$ is in the 21-dimensional symmetric representation, given by

$$(\mathbf{6} \otimes \mathbf{6})_s = \mathbf{21} \quad (163)$$

If we denote the antisymmetric 15 representation by $\Delta_{\alpha\beta}$, we see that the octonionic fields single out the 21 diquark representation at the expense of $\Delta_{\alpha\beta}$. We note that without this color algebra supersymmetry would give antisymmetric configurations as noted by Salam and Strathdee^[38] in their possible supersymmetric generalization of hadronic supersymmetry. Using the nonsingular part of the operator product expansion we can write

$$\tilde{G}_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) = D_{\alpha\beta}(x) + \mathbf{r} \cdot \Delta_{\alpha\beta}(x) \quad (164)$$

The fields $\Delta_{\alpha\beta}$ have opposite parity to $D_{\alpha\beta}$; \mathbf{r} is the relative coordinate at time t if we take $t = t_1 = t_2$. They play no role in the excited baryon which becomes a bilocal system with the 21-dimensional diquark as one of its constituents.

Now consider a three-quark system at time t . The c.m. and relative coordinates are

$$\mathbf{R} = \frac{1}{\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \quad (165)$$

$$\rho = \frac{1}{\sqrt{6}}(2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2) \quad (166)$$

$$\mathbf{r} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \quad (167)$$

giving

$$\mathbf{r}_1 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\rho + \frac{1}{\sqrt{2}}\mathbf{r} \quad (168)$$

$$\mathbf{r}_2 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\rho - \frac{1}{\sqrt{2}}\mathbf{r} \quad (169)$$

$$\mathbf{r}_3 = \frac{1}{\sqrt{3}}\mathbf{R} + \frac{2}{\sqrt{6}}\rho \quad (170)$$

The baryon state must be a color singlet, symmetric in the three pairs (α, x_1) , (β, x_2) , (γ, x_3) . We find

$$(q_\alpha(1)q_\beta(2))q_\gamma(3) = -u_0^*F_{\alpha\beta\gamma}(123) \quad (171)$$

$$q_\gamma(3)(q_\alpha(1)q_\beta(2)) = -u_0F_{\alpha\beta\gamma}(123) \quad (172)$$

so that

$$-\frac{1}{2}\{\{q_\alpha(1), q_\beta(2)\}, q_\gamma(3)\} = F_{\alpha\beta\gamma}(123) \quad (173)$$

The operator $F_{\alpha\beta\gamma}(123)$ is a color singlet and is symmetrical in the three pairs of coordinates. We have

$$F_{\alpha\beta\gamma}(123) = B_{\alpha\beta\gamma}(\mathbf{R}) + \rho \cdot \mathbf{B}'(\mathbf{R}) + \mathbf{r} \cdot \mathbf{B}''(\mathbf{R}) + C \quad (174)$$

where C is of order two and higher in ρ and \mathbf{r} . Because \mathbf{R} is symmetric in \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 , the operator $B_{\alpha\beta\gamma}$ that creates a baryon at \mathbf{R} is totally symmetrical in its flavor-spin indices. In the $SU(6)$ scheme it belongs to the (56) representation. In the bilocal $q-D$ approximation we have $\mathbf{r} = 0$ so that $F_{\alpha\beta\gamma}$ is a function only of \mathbf{R} and ρ which are both symmetrical in \mathbf{r}_1 and \mathbf{r}_2 . As before, \mathbf{B}' belongs to the orbitally excited 70^- representation of $SU(6)$. The totally antisymmetrical (20) is absent in the bilocal approximation. It would only appear in the trilocal treatment that would involve the 15-dimensional diquarks. Hence, if we use local fields, any product of two octonionic quark fields gives a (21) diquark

$$q_\alpha(\mathbf{R})q_\beta(\mathbf{R}) = D_{\alpha\beta}(\mathbf{R}) \quad (175)$$

and any nonassociative combination of three quarks, or a diquark and a quark at the same point give a baryon in the 56^+ representation:

$$(q_\alpha(\mathbf{R})q_\beta(\mathbf{R}))q_\gamma(\mathbf{R}) = -u_0^*B_{\alpha\beta\gamma}(\mathbf{R}) \quad (176)$$

$$q_\alpha(\mathbf{R})(q_\beta(\mathbf{R})q_\gamma(\mathbf{R})) = -u_0B_{\alpha\beta\gamma}(\mathbf{R}) \quad (177)$$

$$q_\gamma(\mathbf{R})(q_\alpha(\mathbf{R})q_\beta(\mathbf{R})) = -u_0 B_{\alpha\beta\gamma}(\mathbf{R}) \quad (178)$$

$$(q_\gamma(\mathbf{R})q_\alpha(\mathbf{R}))q_\beta(\mathbf{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathbf{R}) \quad (179)$$

The bilocal approximation gives the $(35+1)$ mesons and the 70^- baryons with $\ell = 1$ orbital excitation.

A Minimal Scheme

In order to go beyond the Miyazawa supersymmetry we now propose a nonsimple super-Lie algebra that has parabolic structure with the odd part being represented by a symmetric third rank tensor $F_{\alpha\beta\gamma}$ with 56 components and the even part by the 36 $U(6)$ generators L_σ^ρ . The bracket relations are

$$[L_\sigma^\rho, L_\mu^\lambda] = \delta_\mu^\rho L_\sigma^\lambda - \delta_\sigma^\lambda L_\mu^\rho \quad (180)$$

$$\{F_{\alpha\beta\gamma}, F_{\rho\sigma\tau}\} = 0 \quad (181)$$

$$[L_\sigma^\rho, F_{\alpha\beta\gamma}] = \delta_\alpha^\rho F_{\beta\gamma\sigma} + \delta_\beta^\rho F_{\gamma\alpha\sigma} + \delta_\gamma^\rho F_{\alpha\beta\sigma} \quad (182)$$

There is a conjugate superalgebra generated by L_σ^ρ and $F^{\alpha\beta\gamma}$. Note that

$$N = \sum_\rho L_\rho^\rho \quad (183)$$

acts like a number operator and traceless part of L_σ^ρ coincides with the elements of the $SU(6)$ Lie algebra. Hence this minimal superalgebra S has $56 + 35 = 91$ generators. With the addition of color the algebra becomes $SU(3)^c \times S$.

We can immediately construct a matrix representation of this algebra by means of fermionic operators f_α^i that transform like the $(3, 6)$ representation of $SU(3)^c \times SU(6)$ and the conjugate operators \bar{f}_j^β associated with the $(\bar{3}, \bar{6})$ representation. We have

$$\{f_\alpha^i, f_\beta^j\} = 0, \quad \{\bar{f}_i^\alpha, \bar{f}_j^\beta\} = 0 \quad (184)$$

$$\{f_\alpha^i, \bar{f}_j^\beta\} = \delta_j^i \delta_\alpha^\beta \quad (185)$$

Since f_α^i are Grassmann numbers they can be represented by finite matrices belonging to a Clifford algebra. The conjugate operators \bar{f}_α^i are represented by the corresponding Hermitian conjugate matrices. Out of this fermionic Heisenberg basis we construct the following matrices:

$$\Phi_{\alpha\beta\gamma} = \epsilon_{ijk} f_\alpha^i f_\beta^j f_\gamma^k \quad (186)$$

$$M_\sigma^\rho = \bar{f}_i^\rho f_\sigma^i - \frac{1}{6} \delta_\sigma^\rho N \quad (187)$$

where

$$N = \bar{f}_j^\tau f_\tau^j \quad (188)$$

Then $\Phi_{\alpha\beta\gamma}$ and M_α^ρ are the representations of $F_{\alpha\beta\gamma}$ and L_σ^ρ since they satisfy the algebra of Eqs.(180), (181) and (182). If f_α^i are identified with the colored-quark annihilation operators, the odd elements of Φ of the algebra correspond to baryons, while the even elements M correspond to mesons. N represents the number operator.

At this point let us introduce a module that consists of the representations $(\bar{3}, \bar{6})$ and $(\bar{3}, 21)$ of $SU(3)^c \times SU(6)$. They correspond to the operators associated with the antiquarks and diquarks:

$$\bar{Q}_j^\alpha = \bar{f}_j^\alpha, \quad D_{\alpha\beta k} = \epsilon_{ijk} f_\alpha^i f_\beta^j \quad (189)$$

This multiplet is closed under the group $SU(3)^c \times S$. Indeed we have

$$\{\Phi_{\alpha\beta\gamma}, \bar{Q}_k^\rho\} = \delta_\alpha^\rho D_{\beta\gamma k} + \delta_\beta^\rho D_{\gamma\alpha k} + \delta_\gamma^\rho D_{\alpha\beta k} \quad (190)$$

$$[\Phi_{\rho\sigma\tau}, D_{\alpha\beta k}] = 0 \quad (191)$$

$$[M_\rho^\gamma, \bar{Q}_k^\sigma] = \delta_\rho^\sigma \bar{Q}_k^\gamma \quad (192)$$

$$[M_\rho^\gamma, D_{\alpha\beta k}] = \delta_\alpha^\gamma D_{\beta\rho k} + \delta_\beta^\gamma D_{\alpha\rho k} \quad (193)$$

Eqs.(192) and (193) mean that \bar{Q} and D transform respectively, like $(\bar{6})$ and (21) under $SU(6)$. Eqs.(190) and (191) tell us that the odd part of the supergroup maps \bar{Q} into D and D into zero. Letting the corresponding odd parameters be $\eta^{\alpha\beta\gamma}$, we now consider the superalgebra element

$$\Phi = \frac{1}{6}\eta^{\alpha\beta\gamma}\Phi_{\alpha\beta\gamma} \quad (194)$$

We find

$$\delta\bar{Q}_k^\rho = [\Phi, \bar{Q}_k^\rho] = \frac{1}{2}\eta^{\rho\alpha\beta}D_{\alpha\beta k} \quad (195)$$

$$\delta D_{\rho\sigma} = [\Phi, D_{\rho\sigma}] = 0 \quad (196)$$

Neglecting mass and spin differences, the Hamiltonian we wrote down earlier is invariant under the infinitesimal transformations given above.

The baryon-meson system transforms like the adjoint representation of the supergroup S . We find

$$\delta B_{\alpha\beta\gamma} = [\Phi, B_{\alpha\beta\gamma}] = 0 \quad (197)$$

$$\delta M_\sigma^\rho = [\Phi, M_\sigma^\rho] = -\frac{1}{2}\eta^{\rho\alpha\beta}B_{\alpha\beta\sigma} \quad (198)$$

Hence the odd part of S maps mesons into baryons and baryons into zero. If the conjugate supergroup is used, baryons and diquarks are replaced by antibaryons and antidiquarks, while the antiquarks are replaced by quarks.

Note that in this minimal scheme the exotic mesons do not appear. Nevertheless the transformations Eq.(197) and Eq.(198) are sufficient for establishing the equality of the Regge slopes for baryons and mesons.

Because of the form of the cubic function and the quadratic functions of the generators of S in terms of the fermionic annihilation and creation operators, it is natural to introduce the transverse octonion operators as in Eq.(155)

$$f_\alpha = u_i f_\alpha^i, \quad \bar{f}^\beta = -u^{*j} \bar{f}_j^\beta \quad (199)$$

Then we obtain in a general fashion

$$\Phi_{\alpha\beta\gamma} = \frac{1}{2}\{\{f_\alpha, f_\beta\}, f_\gamma\} \quad (200)$$

$$\{\bar{f}^\rho, f_\rho\} = \frac{3}{2}\delta_\sigma^\rho + ie_7 M_\sigma^\rho \quad (201)$$

We also have

$$[f_\alpha, f_\beta] = 0, \quad [\bar{f}^\alpha, \bar{f}^\beta] = 0 \quad (202)$$

$$[f_\alpha, \bar{f}^\beta] = M_\alpha^\beta + \frac{3}{2}ie_7\delta_\alpha^\beta \quad (203)$$

Eqs.(202) and (203) together with the $SU(6)$ Lie algebra of the form Eq.(180) satisfied by the operators M_α^β suggest that M_α^β , f_α , \bar{f}^β , and $ie_7 = (u_0 - u_0^*)$ form an octonionic extension of $SU(6)$ that has $SU(6) \times SU(3)^c$ as a subgroup. It is related to the simple supergroup $SU(6/3)$, but not equivalent to it because of the suppression of the color octet and sextet states in the octonionic products $f_\alpha \bar{f}^\beta$ and $f_\alpha f_\beta$.

The bilocal treatment outlined here also carries over unchanged to the minimal scheme. The (\bar{q}, D) system forms one multiplet $R(1)$ at point x_1 under the supergroup S . The conjugate system (q, \bar{D}) forms a multiplet $R(2)$ at point x_2 under the conjugate group \bar{S} . Then the bilocal system $R(1) \times R(2)$ is expanded as in Eq.(164) and it transforms under the group $S \times \bar{S}$. Its local part with respect to c.m. coordinate x gives the meson-baryon and the meson-antibaryon multiplets $(35, 56^+)$ and $(35, 56^+)$, while the coefficients of the relative coordinate \mathbf{r} give the $\ell = 1$ excitations of mesons in the $(35 + 1)$ representations and the $\ell = 1$ excited baryons in the 70^- representation of $SU(6)$.

In the symmetric approximation, the masses of various states are proportional to the sum of the quark number plus the antiquark number.

Symmetries of the Three Quark System

For the three quark system we can write the generalized form of the two body Hamiltonian

$$\begin{aligned} \tilde{H}_{123} = & \tilde{V}_{12} + \tilde{V}_{23} + \tilde{V}_{31} + \gamma_4^{(1)} \sqrt{[m_1 + \frac{1}{2}S_{12}]^2 + \mathbf{p}_{12}^2} \\ & + \gamma_4^{(2)} \sqrt{[m_2 + \frac{1}{2}S_{12}]^2 + \mathbf{p}_{12}^2} + \gamma_4^{(2)} \sqrt{[m_2 + \frac{1}{2}S_{23}]^2 + \mathbf{p}_{23}^2} \\ & + \gamma_4^{(3)} \sqrt{[m_3 + \frac{1}{2}S_{23}]^2 + \mathbf{p}_{23}^2} + \gamma_4^{(3)} \sqrt{[m_3 + \frac{1}{2}S_{31}]^2 + \mathbf{p}_{31}^2} \\ & + \gamma_4^{(1)} \sqrt{[m_1 + \frac{1}{2}S_{31}]^2 + \mathbf{p}_{31}^2} \end{aligned} \quad (204)$$

where

$$\tilde{V}_{ij} = V_{ij} + \text{spin dependent terms} \quad (205)$$

with $i, j = 1, 2, 3$.

To study the symmetry of this system we introduce fermion annihilation and creation operators for colored quarks f_α^i with $i = 1, 2, 3$ the color index, and $\alpha = 1, \dots, 6$ the flavor index, so that

$$\{f_\alpha^i, f_\beta^j\} = 0, \quad \{\bar{f}_i^\alpha, \bar{f}_j^\beta\} = 0, \quad \{f_\alpha^i, \bar{f}_j^\beta\} = \delta_j^i \delta_\alpha^\beta \quad (206)$$

The diquark transforms like

$$D_{\alpha\beta k} = \epsilon_{ijk} f_\alpha^i f_\beta^k \quad (207)$$

It is in $(\bar{3})$ representation for color, and is in 21 representation for $SU(6)$. The baryons transform like

$$\Phi_{\alpha\beta\gamma} = \epsilon_{ijk} f_\alpha^i f_\beta^j f_\gamma^k \quad (208)$$

It is a color singlet belonging to the 56 representation of $SU(6)$. Color singlet mesons that are in the 35 representation of $SU(6)$ transform like

$$M_\sigma^\rho = \bar{f}_i^\rho f_\sigma^i - \frac{1}{6} \delta_\sigma^\rho N \quad (209)$$

with N being a color singlet and an $SU(6)$ singlet, and is given by

$$N = \bar{f}_j^\tau f_\tau^j \quad (210)$$

$$[M_\sigma^\rho, M_\mu^\lambda] = \delta_\mu^\rho M_\sigma^\lambda - \delta_\sigma^\lambda M_\mu^\rho \quad (211)$$

$$\{\Phi_{\alpha\beta\gamma}, \Phi_{\rho\sigma\tau}\} = 0 \quad (212)$$

$$[M_\sigma^\rho, \Phi_{\rho\sigma\tau}] = \delta_\alpha^\rho \Phi_{\beta\gamma\sigma} + \delta_\beta^\rho \Phi_{\gamma\alpha\sigma} + \delta_\gamma^\rho \Phi_{\alpha\beta\sigma} \quad (213)$$

Now \bar{f}_j^ρ and $D_{\alpha\beta,k}$ form a multiplet that transforms under Eqs.(211), (212), and (213) as follows:

$$\{\Phi_{\alpha\beta\gamma}, \bar{f}_k^\alpha\} = \delta_\alpha^\rho D_{\beta\gamma,k} + \delta_\beta^\rho D_{\gamma\alpha,k} + \delta_\gamma^\rho D_{\alpha\beta,k} \quad (214)$$

$$[\Phi_{\rho\sigma\tau}, D_{\alpha\beta,k}] = 0 \quad (215)$$

$$[M_\rho^\gamma, \bar{f}_k^\sigma] = \delta_\rho^\sigma \bar{f}_k^\gamma \quad (216)$$

$$[M_\rho^\gamma, D_{\alpha\beta,k}] = \delta_\alpha^\gamma D_{\beta\rho,k} + \delta_\beta^\gamma D_{\alpha\rho,k} \quad (217)$$

When the three quark system gets excited, it becomes a $q - D$ system, but retains its symmetry

$$B_{123} = \frac{1}{\sqrt{3}}(q_1 D_{23} + q_2 D_{31} + q_3 D_{12}) \quad (218)$$

and remains in the 56 representation. The 70-representation has opposite parity (for example in $\mathbf{r}_1 - \mathbf{r}_{23}$, etc.).

When in the excited baryon quarks 2 and 3 form a diquark, and we can put

$$\mathbf{p}_{12} = \mathbf{p}_{13} = \mathbf{p} \quad \mathbf{p}_{23} = 0 \quad (219)$$

Then we obtain

$$\begin{aligned} \tilde{H}_{123} \longrightarrow \tilde{H}_{q_1 D_{23}} &\cong 2\tilde{V}_{12} + \gamma_4^{(1)} \sqrt{[m_1 + \frac{1}{2}S]^2 + \mathbf{p}^2} \\ &+ (\gamma_4^{(2)} + \gamma_4^{(3)}) \sqrt{[(m_2 + m_3 + \bar{S}_{23}) + \frac{1}{2}S]^2 + \mathbf{p}^2} \quad (220) \end{aligned}$$

and similarly for the other two configurations obtained by cyclic permutations. Note here that $m_D = m_2 + m_3 + \bar{S}_{23}$ is the mass of the diquark.

Then the superalgebra given by Eqs.(211), (212) and (213) which is symmetric in (123) becomes, for a given $q - D$ configuration the Miyazawa supersymmetry 1 – (23).

When \tilde{H}_{123} is symmetrized it becomes invariant under the discrete group S_3 . Hence its eigenstates are also eigenstates of S_3 , i.e. correspond to representations of S_3 . The splitting of 3 quarks into q and D breaks the S_3 symmetry down to $Z_2 \sim S_2$ which has two eigenstates (associated with ± 1). Iachello^[39] has shown that the 70 representation of $SU(6)$ associated with

$\ell = 1$ is parity doubled while the ground state 56 ($\ell = 0$) is not. This is supported by experiment.

We can now define eight auxiliary octonions of quadratic norm $\frac{1}{2}$ constructed out of the seven imaginary units f_α ($\alpha = 1, \dots, 7$).

$$f_1 = \frac{1}{2}(e_1 + e_4), \quad f_2 = \frac{1}{2}(e_2 + e_5) \quad (221)$$

$$f_3 = \frac{1}{2}(e_3 + e_6), \quad f_4 = \frac{1}{2}(e_1 - e_4) \quad (222)$$

$$f_5 = \frac{1}{2}(e_2 - e_5), \quad f_6 = \frac{1}{2}(e_3 - e_6) \quad (223)$$

$$f_7 = \frac{1}{2}(1 - e_7), \quad f_8 = \frac{1}{2}(1 + e_7) \quad (224)$$

These are intimately related to the eight split octonionic units and their conjugates we discussed above. If we keep one of the f 's fixed, say f_j , then the difference combination $(f_i f_j f_k - f_k f_j f_i)$ is always either zero or equal to another f . We have worked out isomorphisms for such combination rules and they seem to play a fundamental role in dealing with symmetries of three quark systems we discussed, with further applications in multiquark systems. This will be discussed in detail in another publication^[40].

Combinations of f 's also appear naturally within the root systems of groups associated with a magic square and play a profound role in superstring theories and their compactification. Generalization of split units and their implications to projective geometries and internal symmetries is discussed in our recent paper.^[41]

A Colored Supersymmetry Scheme based on $SU(3)^c \times SU(6/1)$

We could go to a smaller supergroup having $SU(6)$ as a subgroup. With the addition of color, such a supergroup is $SU(3) \times SU(6/1)$. The fundamental representation of $SU(6/1)$ is 7-dimensional which decomposes into a sextet and singlet under the spin-flavor group. There is also a 28-dimensional representation of $SU(7)$. Under the $SU(6)$ subgroup it has the decomposition

$$28 = 21 + 6 + 1 \quad (225)$$

Hence, this supermultiplet can accommodate the bosonic antiquark and fermionic quark in it, provided we are willing to add another scalar. Together with the color symmetry, we are led to consider the $(3, 28)$ representation of $SU(3) \times SU(6/1)$ which consists of an antiquark, a quark and a color triplet scalar that we shall call a scalar quark. This boson is in some way analogous to the s quarks. The whole multiplet can be represented by an octonionic 7×7 matrix Z at point x .

$$Z = \mathbf{u} \cdot \begin{pmatrix} \bar{\mathbf{D}}^* & \mathbf{q} \\ i\mathbf{q}^T \sigma_2 & \mathbf{S} \end{pmatrix} \quad (226)$$

Here \mathbf{D}^* is a 6×6 symmetric matrix representing the antiquark, \mathbf{q} is a 6×1 column matrix, \mathbf{q}^T is its transpose and σ_2 is the Pauli matrix that acts on the spin indices of the quark so that, if q transforms with the 2×2 Lorentz matrix L , $\mathbf{q}^T i\sigma_2$ transforms with L^{-1} acting from the right.

Similarly we have

$$Z^c = \mathbf{u}^* \cdot \begin{pmatrix} \mathbf{D} & i\sigma_2 \mathbf{q}^* \\ \mathbf{q}^\dagger & \mathbf{S}^* \end{pmatrix} \quad (227)$$

to represent the supermultiplet with a diquark and antiquark. The mesons, exotic mesons and baryons are all in the bilocal field $Z(1) \otimes Z^c(2)$ which we expand with respect to the center of mass coordinates in order to represent color singlet hadrons by local fields. The color singlets 56^+ and 70^- will then arise as in the earlier section.

Now the $(\bar{D}q)$ system belonged to the fundamental representation of the $SU(6/21)$ supergroup. But Z belongs to the (28) representation of $SU(6/1)$ which is not its fundamental representation. Are there any fields that belong to the 7-dimensional representation of $SU(6/1)$? It is possible to introduce such fictitious fields, such as a 6-dimensional spinor ξ and a scalar a without necessarily assuming their existence as particles. We put

$$\xi = \mathbf{u}^* \cdot \xi, \quad a = \mathbf{u}^* \cdot \mathbf{a} \quad (228)$$

so that both ξ and \mathbf{a} are color antitriplets. Let

$$\lambda = \begin{pmatrix} \xi \\ a \end{pmatrix}, \quad \lambda^c = \begin{pmatrix} \hat{\xi} \\ a^* \end{pmatrix} \quad (229)$$

where

$$\hat{\xi} = \mathbf{u} \cdot (i\sigma_2 \xi^*), \quad a^* = \mathbf{u} \cdot \mathbf{a}^* \quad (230)$$

Consider the 7×7 matrix

$$W = \lambda \lambda^{c\dagger} = \begin{pmatrix} \xi \hat{\xi}^\dagger & \xi a \\ a \hat{\xi}^\dagger & 0 \end{pmatrix} \quad (231)$$

W belongs to the 28-dimensional representation of $SU(6/1)$ and transforms like Z , provided the components of ξ are Grassmann numbers and \mathbf{a} are even (bosonic) coordinates. The identification of Z and W give

$$\mathbf{s} = \mathbf{a} \times \mathbf{a} = 0, \quad \mathbf{q}_\alpha = \xi_\alpha \times \mathbf{a}, \quad D_{\alpha\beta}^* = \xi_\alpha \times \xi_\beta \quad (232)$$

A scalar part in W can be generated by multiplying two different (7-dim) representations. The 56^+ baryons form the color singlet part of the 84-dimensional representation of $SU(6/1)$ while its colored part consists of quarks and diquarks.

Now consider the octonionic valued quark field q_A^i , where $i = 1, 2, 3$ is the color index and A stands for the pair (α, μ) with $\alpha = 1, 2$ being the spin index and $\mu = 1, 2, 3$ the flavor index. (If we have N flavors, $A = 1, \dots, N$). As before

$$q_A = u_i q_A^i = \mathbf{u} \cdot \mathbf{q}_A \quad (233)$$

Similarly the diquark D_{AB} which transforms like a color antitriplet is

$$D_{AB} = q_A q_B = q_B q_A = \epsilon_{ijk} u_k^* q_A^i q_B^j = \mathbf{u}^* \cdot \mathbf{D}_{AB} \quad (234)$$

We note, once again, that because q_A^i are anticommuting fermionic operators, D_{AB} is symmetric in its two indices. The antiquark and antidiquark are represented by

$$\bar{q}_A = \mathbf{u}^* \cdot \bar{\mathbf{q}}_A \quad (235)$$

and

$$\bar{D}_{AB} = \mathbf{u} \cdot \bar{\mathbf{D}}_{AB} \quad (236)$$

respectively. If we have 3 flavors, q_A has 6 components for each color while D_{AB} has 21 components.

At this point let us study the system $(q_A, \bar{\mathbf{D}}_{BC})$ consisting of a quark and an antiquark, both color triplets. There are two possibilities: we can regard the system as a multiplet belonging to the fundamental representation of a supergroup $U(6/21)$ for each color, or as a higher representation of a smaller supergroup. The latter possibility is more economical. To see what kind of supergroup we can have, we imagine that both quarks and diquarks are components of more elementary quantities: a triplet fermion f_A^i and a boson C^i which is a triplet with respect to the color group and a singlet with respect to $SU(2N)$ ($SU(6)$) for three flavors). The f_A^i is taken to have baryon number $1/3$ while C has baryon number $-2/3$. The system

$$q_A^i = \epsilon_{ijk} \bar{f}_A^j \bar{C}^k \quad (237)$$

will be a color triplet with baryon number $1/3$. It can therefore represent a quark. We can write

$$q_A = \mathbf{u} \cdot \mathbf{q}_A = (\mathbf{u}^* \cdot \bar{\mathbf{f}}_A)(\mathbf{u}^* \cdot \bar{\mathbf{C}}) \quad (238)$$

With two anti-f fields we can form bosons that have same quantum numbers as antiquarks:

$$\bar{D}_{AB} = \mathbf{u} \cdot \bar{\mathbf{D}}_{AB} = (\mathbf{u}^* \cdot \bar{\mathbf{f}}_A)(\mathbf{u}^* \cdot \bar{\mathbf{f}}_B) \quad (239)$$

In this case the basic multiplet is $(\mathbf{f}_A, \mathbf{C})$ which belongs to the fundamental representation of $SU(6/1)$ for each color component. The complete algebra to consider is $SU(3) \times SU(6/1)$ and the basic multiplet corresponds to the representation $(3, 7)$ of this algebra. Let

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_6 \\ C \end{pmatrix}, \quad f_A = \mathbf{u} \cdot \mathbf{f}_A, \quad C = \mathbf{u} \cdot \mathbf{C} \quad (240)$$

Also let

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ \vdots \\ f_6^1 \end{pmatrix}, \quad f^2 = \begin{pmatrix} f_1^2 \\ f_2^2 \\ \vdots \\ f_6^2 \end{pmatrix} \quad (241)$$

Combining two such representations and writing $\bar{\mathbf{X}} = \mathbf{F} \times \mathbf{F}^T$ we have

$$\bar{X} = \mathbf{u}^* \cdot \bar{\mathbf{X}} = \begin{pmatrix} f^1 \\ C^1 \end{pmatrix} \begin{pmatrix} f^{2T} & C^2 \end{pmatrix} - \begin{pmatrix} f^2 \\ C^2 \end{pmatrix} \begin{pmatrix} f^{1T} & C^1 \end{pmatrix} \quad (242)$$

Further identifying

$$\mathbf{u}^* \cdot \mathbf{D}_{11} = 2f_1^1 f_1^2, \quad \mathbf{u}^* \cdot \mathbf{D}_{12} = f_1^1 f_2^2 - f_1^2 f_2^1, \quad \text{etc.}, \quad (243)$$

and

$$\mathbf{u}^* \cdot \bar{\mathbf{q}}_1 = f_1^1 C^2 - f_1^2 C^1, \quad \mathbf{u}^* \cdot \bar{\mathbf{q}}_2 = f_2^1 C^2 - f_2^2 C^1, \quad \text{etc.}, \quad (244)$$

we see that \bar{X} has the structure

$$\bar{X} = \mathbf{u}^* \cdot \bar{\mathbf{X}} = \mathbf{u}^* \cdot \begin{pmatrix} \mathbf{D}_{11} & \dots & \mathbf{D}_{16} & \bar{\mathbf{q}}_1 \\ \mathbf{D}_{12} & \dots & \mathbf{D}_{26} & \bar{\mathbf{q}}_2 \\ \mathbf{D}_{13} & \dots & \mathbf{D}_{36} & \bar{\mathbf{q}}_3 \\ \mathbf{D}_{14} & \dots & \mathbf{D}_{46} & \bar{\mathbf{q}}_4 \\ \mathbf{D}_{15} & \dots & \mathbf{D}_{56} & \bar{\mathbf{q}}_5 \\ \mathbf{D}_{16} & \dots & \mathbf{D}_{66} & \bar{\mathbf{q}}_6 \\ -\bar{\mathbf{q}}_1 & \dots & -\bar{\mathbf{q}}_6 & 0 \end{pmatrix} \quad (245)$$

or,

$$(3, 7) \times (3, 7) = (\bar{3} \times 27) \quad (246)$$

The 27 dimensional representation decomposes into $21 + \bar{6}$ with respect to its $SU(6)$ subgroup.

Consider now an antiquark-diquark system at point $x_1 = x - \frac{1}{2}\xi$ and another quark-antidiquark system at point $x_1 = x + \frac{1}{2}\xi$; Hence we take the direct product of $X(x_1)$ and $X(x_2)$. In other words

$$(D_{AB}(x_1), \bar{q}_D(x_1)) \otimes (\bar{D}_{EF}(x_2), q_C(x_2)) \quad (247)$$

consisting of the pieces

$$H(x_1, x_2) = \begin{pmatrix} \bar{q}_D(x_1)q_C(x_2) & D_{AB}(x_1)q_C(x_2) \\ \bar{q}_D(x_1)\bar{D}_{EF}(x_2) & D_{AB}(x_1)\bar{D}_{EF}(x_2) \end{pmatrix} \quad (248)$$

The diagonal pieces are bilocal fields representing color singlet $1 + 35$ mesons and $1 + 35 + 405$ exotic mesons respectively with respect to the subgroup $SU(3)^c \times SU(6)$ of the algebra. The off diagonal pieces are color singlets that are completely symmetrical with respect to the indices (ABC) and (DEF) . They correspond to baryons and antibaryons in the representations 56 and $\bar{56}$ respectively of $SU(6)$.

We can write $F_{ABC} = -\frac{1}{2}\{D_{AB}, q_C\}$ so that

$$D_{AB}q_C = (u_1^*(q_A^2q_B^3 + q_B^2q_A^3) + u_2^*(q_A^3q_B^1 + q_B^3q_A^1) + u_3^*(q_A^1q_B^2 + q_B^1q_A^2)) \times (u_1q_C^1 + u_2q_C^2 + u_3q_C^3) \quad (249)$$

becomes

$$D_{AB}q_C = -u_0^*((q_A^2q_B^3 + q_B^2q_A^3)q_C^1 + (q_A^3q_B^1 + q_B^3q_A^1)q_C^2 + (q_A^1q_B^2 + q_B^1q_A^2)q_C^3) \quad (250)$$

and similarly

$$q_CD_{AB} = -u_0((q_A^2q_B^3 + q_B^2q_A^3)q_C^1 + (q_A^3q_B^1 + q_B^3q_A^1)q_C^2 + (q_A^1q_B^2 + q_B^1q_A^2)q_C^3) \quad (251)$$

Since $u_0 + u_0^* = 1$, we have

$$F_{ABC} = -\frac{1}{2}\{D_{AB}, q_C\} = (q_A^1q_B^2 + q_B^1q_A^2)q_C^3 + (q_A^2q_B^3 + q_B^2q_A^3)q_C^1 + (q_A^3q_B^1 + q_B^3q_A^1)q_C^2 \quad (252)$$

which is completely symmetric with respect to indices (ABC) , corresponding to baryons.

In the limit $x_2 - x_1 = \xi \longrightarrow 0$, H can be represented by a local supermultiplet with dimension $2 \times 56 + 2(1 + 35) + 405 = 589$ of the original algebra. This representation includes 56 baryons, antibaryons, mesons and $q^2\bar{q}^2$ exotic mesons.

Transformation Properties

Since $F = \mathbf{u} \cdot \mathbf{F}$ consists of three 7-dimensional representations of $SU(6/1)$ we have

$$\delta F = Z F \quad (253)$$

where $Z \in SU(6/1)$,

$$F = \begin{pmatrix} f \\ C \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_6 \\ C \end{pmatrix} \quad (254)$$

Hence Z is a super antihermitian color singlet:

$$Z = \begin{pmatrix} iH & \eta \\ i\eta^\dagger & i\omega \end{pmatrix} \quad (255)$$

with $H = H^\dagger$, $\omega = \omega^*$, $Str Z = 0$ (Str = supertrace), ($\omega = tr H$), and

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_6 \end{pmatrix}, \quad \{\eta_\alpha, \eta_\beta\} = 0 \quad (256)$$

H is an antihermitian 6×6 matrix.

Then by taking supertransposed quantities (sT = supertransposed)

$$\delta F^{sT} = \delta F^T = F^T Z^{sT} \quad (257)$$

we have

$$\begin{pmatrix} \delta \mathbf{f} \\ \delta \mathbf{C} \end{pmatrix} = \begin{pmatrix} iH & \eta \\ i\eta^\dagger & i\omega \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{C} \end{pmatrix} \quad (258)$$

$$\begin{pmatrix} \delta \mathbf{f}^T & \delta \mathbf{C} \end{pmatrix} = \begin{pmatrix} iH^* & -i\eta^* \\ \eta^T & i\omega \end{pmatrix} \quad (259)$$

We also have

$$\bar{X} = F \quad F^T = F \quad F^{sT} \quad (260)$$

so that

$$\delta \bar{X} = Z \quad \bar{X} + \bar{X} \quad Z^{sT} \quad (261)$$

Now writing

$$\bar{X} = \begin{pmatrix} \bar{D} & q \\ -q^T & 0 \end{pmatrix}, \quad (D = D^T) \quad (262)$$

under $U(6/21)$, δX gives:

$$\delta \bar{D} = i(H\bar{D} + \bar{D}H^T) - (\eta q^T - q\eta^T) \quad (263)$$

$$\delta q = i(H + \omega)q - \bar{D}\eta^* \quad (264)$$

For supertransformation $SU(6/1)/U(6)$, the change in \bar{D} and q are

$$\delta \bar{D} = q\eta^T - \eta q^T, \quad \delta q = -\bar{D}\eta^* \quad (265)$$

Since

$$\delta f = Zf \quad (266)$$

or, in component form

$$\delta f_A = Z_{AB}f_B \quad (267)$$

where $(A, B = 0, 1, \dots, 6)$ with $f_0 = C$, we have

$$\delta f^{sT*} = \delta f^\dagger = f^\dagger \quad Z^{*sT} \quad (268)$$

If we define g by

$$g = \begin{pmatrix} I & 0 \\ 0 & i \end{pmatrix} \quad (269)$$

then

$$\delta(f^\dagger \quad g \quad f) = f^\dagger(gZ + Z^{*sT}g)f \quad (270)$$

It is easy to show $gZ + Z^{*sT}g = 0$ so that $\delta(f^\dagger g f) = 0$. If we look at $U(2/1)$ parts

$$\delta F = \begin{pmatrix} iH & 0 \\ 0 & i\omega \end{pmatrix} \quad (271)$$

giving

$$\delta f = iHf, \quad \delta C = i\omega C \quad (272)$$

and

$$\delta F^\dagger = F^\dagger \begin{pmatrix} -iH & 0 \\ 0 & -i\omega \end{pmatrix} \quad (273)$$

Since

$$\delta f^\dagger = -if^\dagger H, \quad \delta C^* = -i\omega C^* \quad (274)$$

we obtain

$$\delta(f^\dagger f) = 0 \quad (275)$$

and

$$\delta(C^* C) = 0 \quad (276)$$

Similarly

$$\delta F = \begin{pmatrix} 0 & \eta \\ i\eta^\dagger & 0 \end{pmatrix} F \quad (277)$$

gives

$$\delta f = \eta C, \quad \delta C = i\eta^\dagger f \quad (278)$$

and using

$$\delta f^\dagger = C^* \eta^\dagger, \quad \text{and} \quad \delta C^* = -i\eta^T f^* = if^\dagger \eta \quad (279)$$

we arrive at

$$\delta(if^\dagger f - C^\dagger C) = 0 \quad (280)$$

Defining $\bar{f} = i f^\dagger$, we have

$$\delta(\bar{f}f - C^\dagger C) = 0 \quad (281)$$

If we now define

$$h_{AB\dot{A}\dot{B}} = (\bar{X})_{AB}(x)(X)_{\dot{A}\dot{B}}(x) \quad (282)$$

where $A, B = 0, 1, \dots, 6$ as before, then the subset $h_{0b0\dot{b}}$ antisymmetric in the first and last pairs of indices would describe $q\bar{q}$ mesons, $h_{ab\dot{a}\dot{b}}$ symmetric in first and last pairs of indices would describe $q^2\bar{q}^2$ exotic mesons, $h_{0b\dot{a}\dot{b}}$ antisymmetric in the first and symmetric in the last pair of indices would describe q^2q baryons, and $h_{ab0\dot{b}}$ symmetric in the first and antisymmetric in the last pair of indices would describe $\bar{q}^2\bar{q}$ antibaryons.

Aside from $h_{AB\dot{A}\dot{B}}$ describing baryons, antibaryons, mesons and exotics, this algebra can be extended to include preons F_A , antipreons $F_{\dot{A}}$, X_{AB} describing (\bar{q}^2q) , and $X_{\dot{A}\dot{B}}$ describing $(q^2\bar{q})$. Gauge bosons and gauginos can be in the adjoint representation V_{AB} .

We note that, since

$$(Z_{AB})^* = (Z^*)_{\dot{A}\dot{B}} \quad (283)$$

we have

$$\delta F_A = Z_{AB} F_B \text{ and } \delta F_{\dot{A}} = Z_{AB}^* F_{\dot{B}} \quad (284)$$

Further applications of these extended algebraic structures and their implication for the construction of supersymmetric meson baryon lagrangians will be dealt in another publication.

Comments on Relativistic Formulation Through the Spin Realization of the Wess-Zumino Algebra

It is possible to use a spin representation of the Wess-Zumino algebra to write first order relativistic equations for quarks and diquarks that are invariant under supersymmetry transformations. In this section we briefly deal with such Dirac-like supersymmetric equations and with a short discussion of experimental possibilities for the observation of the diquark structure and exotic $\bar{D}-D = (\bar{q}\bar{q})(qq)$ mesons. For a very nice discussion of the experimental situation we refer the reader to a recent papers by Anselmino, et.al.^[9], and by Klempf^[42].

There is a spin realization of the Wess-Zumino super-Poincaré algebra

$$[p_\mu, p_\nu] = 0, \quad [D_\alpha, p_\mu] = 0 \quad (285)$$

$$[\bar{D}_{\dot{\beta}}, p_\mu] = 0, \quad [D^\alpha, \bar{D}^{\dot{\beta}}] = \sigma_\mu^{\alpha\dot{\beta}} p^\mu \quad (286)$$

with p_μ transforming like a 4-vector and $D^\alpha, \bar{D}^{\dot{\beta}}$ like the left and right handed spinors under the Lorentz group with generators $J_{\mu\nu}$.

We also note that

$$[J_{\mu\nu}, p_\lambda] = \delta_{\mu\nu} p_\lambda - \delta_{\nu\lambda} p_\mu \quad (287)$$

and

$$[J, J] = J \quad (288)$$

The finite non unitary spin realization is in terms of 4×4 matrices for $J_{\mu\nu}$ and p_ν

$$J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} = \frac{1}{4i} [\gamma_\mu, \gamma_\nu] \quad (289)$$

$$J_{\mu\nu}^L = \frac{1 - \gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} = \Sigma_{\mu\nu}^L \quad (290)$$

$$p_\mu = \Pi_\mu^L = \frac{1 - \gamma_5}{2} \gamma_\mu \quad (291)$$

Introducing two Grassmann numbers θ_α ($\alpha = 1, 2$) that transforms like the components of a left handed spinor and commute with the Dirac matrices γ_μ , we have the representation

$$D_\alpha = \Delta_\alpha = \frac{\partial}{\partial \theta_\alpha} \quad (292)$$

$$\bar{D}^{\dot{\beta}} = \bar{\Delta}^{\dot{\beta}} = \theta_\alpha \sigma_\mu^{\alpha\dot{\beta}} \Pi_\mu^L \quad (293)$$

Such a representation of the super-Poincaré algebra acts on a Majorana chiral superfield

$$S(x, \theta) = \psi(x) + \theta_\alpha B^\alpha(x) + \frac{1}{2} \theta_\alpha \theta^\alpha \chi(x). \quad (294)$$

Here ψ and χ are Majorana superfields associated with fermions and B^α has an unwritten Majorana index and a chiral spinor index α , so that it represents a boson.

Note that the sum of the two representations we wrote down is also a realization of the Wess-Zumino algebra.

On the other hand we have the realization of p_μ in terms of the differential operator $-i\partial_\mu = -i\frac{\partial}{\partial x^\mu}$. In the Majorana representation, the operator $\gamma_\mu\partial_\mu = i\gamma_\mu p_\mu$ is real, and $\psi = \psi^c = \psi^*$. Let us now define ψ_L and ψ_R by

$$\psi_L = \frac{1}{2}(1 + \gamma_5)\psi \quad (295)$$

and

$$\psi_R = \frac{1}{2}(1 - \gamma_5)\psi = \psi_L^* \quad (296)$$

The free particle Dirac equation can now be written as

$$\Pi_\mu^L \partial_\mu \psi_L = m\psi_L^* \quad (297)$$

or

$$\Pi_\mu p^\mu \psi_L = -im\psi_L^* \quad (298)$$

We can introduce left and right handed component fields

$$S_L = \frac{1}{2}(1 + \gamma_5)S \quad (299)$$

$$S_R = \frac{1}{2}(1 - \gamma_5)S = S_L^* \quad (300)$$

so that Eq.(297) generalizes to the superfield equation

$$\Pi_\mu^L \partial_\mu S_L = mS_L^* \quad (301)$$

or,

$$\Pi_\mu p^\mu S_L = -imS_L^* \quad (302)$$

Now consider the supersymmetry transformation

$$\delta S_L = (\xi^\alpha \Delta_\alpha + \bar{\xi}_{\dot{\beta}} \bar{\Delta}^{\dot{\beta}})S_L = \Xi S_L \quad (303)$$

This transformation commutes with the operator $\Pi_\mu^L \partial_\mu$ so that

$$\Pi_\mu^L \partial_\mu (S_L + \delta S_L) = m(S_L + \delta S_L)^*. \quad (304)$$

If ψ_L is a left handed quark and $B^\alpha(x)$ an antiquark with the same mass as the quark, Eq.(303) provides a relativistic form of the quark antiquark symmetry which is in fact broken by the quark-diquark mass difference. The scalar supersymmetric potential is introduced through $m \longrightarrow m + V_s$ as before and Eq.(301) remains supersymmetric. By means of this formalism, it is possible to reformulate the treatments given in the earlier sections in first order relativistic form.

To write equations in the first order form, we consider V and Φ given in terms of the boson fields by

$$V = i\gamma_\mu V_\mu - \frac{1}{2}\sigma_{\mu\nu} \quad (305)$$

and

$$\Phi = i\gamma_5 \phi + i\gamma_5 \gamma_\mu \phi_\mu. \quad (306)$$

In the Majorana representation

$$V^* = -V, \quad \text{and} \quad \Phi = \Phi^* \quad (307)$$

We now define the left and right handed component fields by

$$V_L = \frac{1 - \gamma_5}{2} V \quad (308)$$

and

$$V_R = \frac{1 + \gamma_5}{2} V \quad (309)$$

so that

$$V_R = -V_L^* \quad (310)$$

with

$$V_L = i\frac{1 - \gamma_5}{2} \gamma_\mu V_\mu - \frac{1 - \gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} V_{\mu\nu} \quad (311)$$

and making use of Eq.(291) we have

$$V_L = i\Pi_\mu^L V_\mu - \frac{1-\gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} V_{\mu\nu} \quad (312)$$

Noting that for any a_μ and b_ν we can write

$$\Pi_\mu \Pi_\nu^* a_\mu b_\nu = \frac{1-\gamma_5}{2} \gamma_\mu \frac{1+\gamma_5}{2} \gamma_\nu a_\mu b_\nu = \frac{1-\gamma_5}{2} \gamma_\mu \gamma_\nu a_\mu b_\nu \quad (313)$$

and incorporating the definition of $\sigma_{\mu\nu}$

$$\frac{1}{2}(\Pi_\mu \Pi_\nu^* - \Pi_\nu \Pi_\mu^*) = \frac{1-\gamma_5}{2} i\sigma_{\mu\nu} \quad (314)$$

in Eq.(312) leads to

$$V_L = i\Pi_\mu^L V_\mu + \frac{i}{2}(\Pi_\mu \Pi_\nu^* - \Pi_\nu \Pi_\mu^*) V_{\mu\nu} \quad (315)$$

Letting $\Sigma_{\mu\nu}^L = \Pi_\mu \Pi_\nu^* - \Pi_\nu \Pi_\mu^*$, Eq.(315) reads

$$V_L = i\Pi_\mu^L V_\mu + \frac{i}{2} \Sigma_{\mu\nu}^L V_{\mu\nu} \quad (316)$$

We now have a first order equation:

$$\Pi_\mu^L \partial_\mu V_R = m V_L \quad (317)$$

Similarly

$$\Pi_\mu^R \partial_\mu V_L = m V_R \quad (318)$$

Therefore

$$\Pi_\mu^L \Pi_\mu^R \partial_\mu \partial_\nu V_L = m \Pi_\mu^L \partial_\mu V_R = m^2 V_L \quad (319)$$

which after substitution of Eq.(317) gives

$$\square V_L = m^2 V_L \quad (320)$$

For the Φ part we can write

$$\Pi_\mu^L \partial_\mu \Phi_R = m \Phi_L \quad (321)$$

with

$$\Phi_L = \frac{1 - \gamma_5}{2} \Phi \quad (322)$$

and

$$\Phi_R = \frac{1 + \gamma_5}{2} \Phi \quad (323)$$

Clearly,

$$\Phi_R = \Phi_L^*. \quad (324)$$

and above procedure can now be repeated for the Φ fields.

Further Outlook

We have shown that the quark model with potentials derived from QCD, including the quark-diquark model for excited hadrons, gives mass formulae in very good agreement with experiment and goes a long way in explaining the approximate symmetries and supersymmetries of the hadronic spectrum, including the symmetry-breaking mechanism. For heavy quarks the non-relativistic approximation can be used so that the potential models for the spectra of charmonium^[43] and the $b\bar{b}$ system^[44] are even simpler. In this approach gluons are eliminated leaving quarks interacting through potentials.

It is also possible to take an opposite approach by eliminating quarks as well as gluons, leaving only an effective theory that involves mesons (quark bound states) and baryons as collective excitations (solitons) of the mesonic field. This is the way pioneered by Skyrme^[45] and revived in recent years^[46]. The simplest model uses an $SU(2) \times SU(2)$ nonlinear chiral model involving pi-mesons as originally formulated by Skyrme in 1958. In 1960 Skyrme added a fourth order term to stabilize the soliton, so that it could be interpreted as a nucleon. The model was enriched by the Syracuse school^[47] and by Witten and his collaborators^[48]. More recently vector mesons were added, improving the predictions^[49]. In the original model, mass formulae were obtained only for a tower of $I = S$ excitations. Recently $SU(4)$ symmetry^[50] and in principle $SU(6)$ symmetry were incorporated and through fermionic quantization of the soliton the tower excitations were chopped off, leading to a more realistic model^[51].

However, even in these improved versions the mass relations are not nearly as good as in the potential quark model when the correct pion decay constant

is used^[52]. There is also an overall positive contribution to hadronic masses that is not easily disposed of, although there are some recent attempts to deal with the problem^[53]. In the absolute scale baryon masses are about 20% too high and vector mesons enter the theory on a very different footing than pseudoscalar mesons, making it very difficult to relate their masses.

A skyrme model that can compete with the potential quark model is still hidden in the future. Partial solution to improved Skyrme model was given in our recent paper,^[55] suggesting a strategy towards building a full fledged quantitatively viable skyrmionic description of hadronic physics.

A better understanding of hadron masses through the QCD theory of quarks and gluons has shifted the mass problem from the hadronic level to the level of the elementary constituent quarks, leptons, weak bosons and Higgs bosons of the standard model. Since we now know that there are three generations of fundamental fermions with light left handed neutrinos, the development of a theory of masses for the finite system has become a burning fundamental problem. What is needed is the equivalent of a Gell-Mann-Okubo^[54] formula for the six quarks and for the leptons. A group theoretical treatment would have to involve a horizontal group associated with the three generations, namely a new $SU(3)$.

In order to approach the fundamental mass problem in imitation of hadron masses we would have to introduce new primary particles (preons) that would yield quarks and leptons as bound states. But then it would be very difficult to understand why the fundamental fermions of the standard model are finite in number.

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